The Volatility Premium

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Abstract

The difference average risk-neutral and physical volatility substantial and translates into a large return premium for sellers of index options. This paper studies a general equilibrium model based on long-run risk in an effort to explain the premium. In estimating the model using data on stock returns and volatility (VIX), the model captures the premium and also the large negative correlation between shocks to volatility and stock prices. Numerical simulations verify that writers of index options earn high rates of return in equilibrium and that the return patterns are similar to that seen in the S&P 500 index options data.

JEL classification: G12, G13, C15.

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1 Introduction

> *Long-Term shorted options at prices that implied a market volatility of 19%. As options prices rose, Long-Term continued to sell. Other firms sold in tiny amounts. Not Long-Term. It just kept selling. .. Eventually they had a staggering $ 40 million riding on each percentage point change in the equity volatility in the United States and an equivalent amount in Europe - perhaps a fourth of the overall market. Morgan Stanley coined a nickname for the fund: the Central Bank of Volatility.*

Roger Lowenstein, "When Genius Failed: The Rise and Fall of Long-Term Capital Management"

The practice of selling volatility is a favorite among hedge funds. Traditionally, investors who "sell volatility" typically take a simultaneous short position in put and call options (straddles). Such positions have a net positive return if the underlying stock price moves very little before option expiration. Conversely, the investor loses money if the price increases or decreases a lot prior to expiration. It yields a positive average return over time if the option implied volatility systematically exceeds actual price volatility. Recent market innovations such as variance swaps and futures on the VIX volatility index allow investors to buy and sell volatility like any other asset. For example, a variance swap pays the difference between "realized variance," defined to be the average squared daily return, and the squared VIX index, allowing the investor to bet directly on the difference between physical, realized stock price variation and the variation implicit in options prices (the VIX index).

It is well known that on average, the implied volatility of index options exceeds the unconditional annualized standard deviation of the underlying index. For example, the VIX index, which gives a model-free (non-parametric) option implied estimate of the volatility of the S&P 500, averages about 19% between 1990 and 2007. The unconditional annualized standard deviation of the S&P 500 is only about 15.7%. The 3.3% difference between option implied and realized standard deviation suggests that *ex-ante*, the premium for writing options on the S&P 500, is substantial. For example, if we consider a one month maturity at-the-money option, an option priced at 19% implied volatility is about 18% more expensive than one priced at 16% implied volatility. In a Black-Scholes world, this 18% translates into pure arbitrage profits for writers of options because the writer can perfectly hedge a position that pays the difference between the market price and the theoretical price. In the real world, obviously, these gains cannot be pocketed risk free. Rather, a short position in volatility entails substantial risk since the volatility itself changes randomly over time. Still, empirical evidence suggests that the average returns generated by issuers of options are substantial and yield risk-reward ratios that far exceed those of other asset classes including broad equity indices such as the CRSP or the S&P 500.

Indeed, several papers have assessed the size of the volatility premium and the risk rewards offered to writers of index options. Coval and Shumway (2001) report monthly Sharpe ratios of about 0.3 (corresponding to annualized numbers of about 1) to investors who write crash-protected straddles[1]. Driessen and Maenhout (2006) examine US equity index options from 1992 to 2001 and find that various options strategies give annualized Sharpe ratios of about 0.72. Eraker (2007b) reports an annualized Sharpe ratio of 0.45 from selling all options available. While Sharpe ratios in the 0.45 to 1 range may seem persuasive, there is also considerable uncertainty associated with the numbers, as the empirical studies rely on relatively short sampling periods. On the other hand, the crash-protected straddle strategy in Coval and Shumway requires the purchase of out-of-the-money put options which is an expensive way to hedge downside risk. Of course, the relatively high prices of out-of-the-money puts has motivated much of the research on generalized options pricing models. Indeed, much of this work has focused on developing no-arbitrage models which can explain the steepness of the Black-Scholes implied volatility "smile" which again is indicative of the high price of out-of-the-money puts.

In recent work Brodie, Chernov, and Johannes (2007b) point out that put options have large negative betas which in turn yield large negative expected rates of return if the CAPM holds. Without considering the volatility premium, they show that writing 6% out-of-the-money puts earns an average monthly rate of return of -22.6 % under the standard Black-Scholes and classic CAPM assumptions if the annual Sharpe ratio in the stock market is 0.06/0.15=0.4 (see Brodie, Chernov, and Johannes Table 4). The large negative average return documented in their paper is purely a risk premium for directional

[1]The crash protection is accomplished by buying a put that is ten percent out of the money for each straddle, effectively capping the loss potential at ten percent. The annualization of Coval and Shumway's monthly numbers is accomplished by multiplying by $\sqrt{12}$.

stock price exposure. This risk premium comes from the simple fact that OTM puts have astronomically large market betas resulting from the fact that the beta of a put option approaches negative infinity as the strike approaches zero. Brodie et al. do not consider adjustments for the directional price exposure which is easily incorporated by delta hedging. In other words, their modeling framework provides zero risk premium to zero-delta option portfolios, such as at-the-money straddles.

Bakshi and Kapadia (2003) study gains from delta-hedged puts and calls over various maturities and strikes. They find significantly negative premia across various maturity and strike categories. In particular, they report that out-of-the-money put options lose, on average, between 82 and 91% of their initial value. Returns on out-of-the-money puts that were between four and six percent OTM averaged negative 95% and 58% percent respectively in Bondarenko (2003). Eraker (2007b) studies an elaborate hedging scheme and finds annualized Sharpe ratios as high as 1.6. Finally, it should be noted that in no way do Sharpe ratios actually exhaust the real risks involved in selling volatility because the returns from volatility-based strategies are highly non-gaussian such that an investor without mean-variance utility is likely to require substantial premia for tail-risks involved in options strategies.

This paper seeks to find an equilibrium explanation for the volatility premium. In our quest for a rationalization of the premium, consider first the simplest of equilibrium models - the CAPM. It is well documented that the volatility of the S&P 500 is massively negatively correlated with the S&P 500 returns themselves. Estimates of this correlation over different sampling intervals typically range from -0.7 to -0.9. Considering that the volatility of the relative changes in the VIX volatility index is about 0.05, five times that of the S&P itself, an asset whose returns move one-to-one with relative changes in the VIX would have a market beta in the range of -3.5 to -4.5. The CAPM, obviously, prescribes a very sizable, negative risk premium to such an asset. For example, with a beta of -4.5 and a 7% annual market risk premium, the risk premium for selling volatility is 31.5% according to the CAPM. While this is a large number, it is still much smaller than the 83% annual returns reported from short options positions in Driessen and Maenhout (2006) or the 160% in Coval & Shumway (2001). Indeed, Bondarenko (2003) computes CAPM betas for the option returns and finds that the model produces large statistically significant alphas and explains very little of the average option returns.

The problem with our back-of-the-envelope CAPM computation is that the model does not really apply in its simplest form to an economy with randomly changing volatility, as is assumed here. Moreover, while I have argued that a negative correlation between volatility and returns exists, it is equally important to understand why this correlation exists. To this end this paper uses a model in which the negative return-volatility relation results from an endogenous negative price response to increases in economic uncertainty. The size of the correlation, therefore, depends on investors' preferences towards uncertainty. This model is not a traditional CAPM model where volatility has a negative market risk premium because it is assumed to be negatively correlated with market returns. Rather, the direction of causality is the opposite: the aggregate market return has a high positive risk premium because it correlates negatively with volatility shocks.

This paper studies the volatility premium through the general equilibrium model proposed in Eraker and Shaliastovich (2008). This is a simple model of an endowment economy where uncertainty about future economic growth fluctuates over time. Incorporating the random time variation in the macro-economic uncertainty is a key element of the model. Stock prices in this economy are obtained as the present value of future dividends, discounted using an endogenously defined equilibrium stochastic discount factor. Since expected future dividends do not change when uncertainty about the future does, an increase (decrease) in uncertainty leads to an endogenous decrease (increase) in stock prices. This captures the aforementioned negative return-volatility correlation.

The Eraker-Shaliastovich model is based on a long-run risk equilibrium formulation. Long-run risk models, as pioneered in Bansal and Yaron (2004), are based upon the idea that shocks having multi-period, long-run effects are priced in equilibrium when investors have preferences over the timing of uncertainty resolution which differ from their intertemporal elasticity of consumption substitution.

Separating the two, as is the case in the recursive preference structure of Kreps-Porteus-Epstein-Zin (Kreps and Porteus (1978) , Epstein and Zin (1989), and Duffie and Epstein (1992)) is crucial for these long-run effects to occur. By contrast, standard CRRA preferences produce zero risk premia for all shocks that do not directly affect consumption. CRRA preferences do not generate risk premia that increase with the persistence of volatility or other state-variables. In fact, imposing the parametric constraints that yield a CRRA preference structure onto the KPEZ preferences, long-run risk models including the one presented here produce a zero market price of volatility risk, and thus a zero volatility premium.

Though the equilibrium model studied in this paper resembles the Bansal-Yaron model, there are several important differences. First, this model is based on a continuous time formulation. It does not have a stochastic persistent growth rate of consumption like the BY model. There are two priced shocks in the model: shocks to consumption growth and shocks to the volatility of consumption growth. The shocks to volatility can either be small (Brownian motion) or potentially large, causing the volatility path to be discontinuous (jump). The possibility of large shocks to economic uncertainty helps explain the sizable risk premia associated with volatility.

In concurrent independent work, Yaron and Drechsler (2010) and Drechsler (2008) study option pricing under long-run risk specifications and discuss the volatility premium in particular. Drechler and Yaron conclude that their model can replicate the unconditional volatility risk premium in seen in the data when adding jumps to the volatility process, as in Eraker and Shaliastovich. These papers do not address whether their models are capable of generating an endogenous volatility risk premium large enough to explain the substantial negative average returns on options, or whether the model can generate the leverage effect in equilibrium. I am unaware of any existing model capable of generating a sample correlation of -0.72 between changes in volatility and returns simply by endogenizing the price response to changes in uncertainty. For example, Wu (2001) proposes a model based on partial equilibrium which produces a -0.61 correlation, but this is obtained by assuming that dividends and volatility are exogenously negatively correlated. Using calibration to monthly data consumption equity and bond data, Bansal & Yaron (2004) model produces a volatility-return correlation of -.32.

There are many papers that consider equilibrium based on time-separable preferences. Classic articles on this issue include Merton (1973), Breeden (1979), Duffie and Zame (1989), and Cox, Ingersoll, and Ross (1985) among others. In a precursor to the Bansal-Yaron analysis, Campbell (1993) studies KPEZ preferences in the context of state-variables driven by VARs. Preferences can be inferred from state-prices implicit in derivatives prices as in Breeden and Litzenberger (1978) and Aït-Sahalia and Lo (2000) who provide non-parametric estimates of preferences from options. Bates (2006) considers equilibrium in the context of agents with a particular aversion toward downside risk (crashes) in order to explain the put premium. Liu, Pan, and Wang (2005) study recursive preferences obtained under uncertainty aversion and show that model ambiguity can explain the large premium on put options. Other papers which consider KPEZ preferences in the context of options pricing include Garcia, Luger, and Renault (2003) and Benzoni, Collin-Dufresne, and Goldstein (2005). Bansal, Dittmar, and Kiku (2005), Bansal, Gallant, and Tauchen (2007) and Campbell and Beeler (2010), and Ferson, Nallaready, and Xie (2010) present empirical evidence on the performance of long-run risk models.

A large part of the finance literature is concerned with developing and estimating no-arbitrage models of asset prices. A sizable literature exists on developing and estimating such models for options pricing. Semi-analytical pricing models were developed in Stein and Stein (1991) and Heston (1993) (stochastic volatility), and Bates (1996) (jumps). Bakshi, Cao, and Chen (1997), Bates (2000), and Eraker (2004) consider empirical tests of such models. A survey of this literature can be found in Singleton (2006). Explaining the volatility premium with a no-arbitrage model is easy because no-arbitrage models essentially allow market prices of risk to be free parameters. Since the premium is a function of the market price of risk, it is possible to take almost any no-arbitrage model with stochastic volatility and assign a market price of risk large enough to generate a sufficiently large difference between the option-implied and observed volatility. By contrast, therefore, this paper seeks to find an equilibrium interpretation of the premium. This is much more difficult, because the market prices of risk in an equilibrium model are intimately tied to risk preferences as well as the dynamics describing the exogenous (macro) quantities in the economy. The challenge, therefore, is to find a model specification coupled with parameter estimates that imply asset return moments that are broadly consistent with the equity premium and the return variability, as well as the return-volatility correlation and the volatility premium.

This paper uses a novel estimation approach based on likelihood inference. Since the theoretical model implies a linear relationship between economic uncertainty and implied options volatility, estimation is conducted using observed stock prices and implied volatility (VIX index). Using VIX data allows us to identify all the parameters that determine the dynamic behavior of volatility. By constraining mean consumption growth and consumption volatility to equal that which is observed in consumption data, the remaining parameters, notably preference parameters, are inferred from returns and volatility data using Bayesian MCMC likelihood inference.

The empirical results are as follows: the volatility premium averages 3.3 percent in annualized standard deviation units and 1.5 in variance units in the data. The equilibrium model produces a premium of 3.8 percent in standard deviation units and 1.47 in variance units over the sample. These differences between the data and the model are statistically insignificant. The equilibrium model also produces an endogenous correlation between changes in volatility and stock returns (the so-called "leverage effect" or asymmetric volatility) of -0.66, which compares to -0.72 in the data. This difference is again statistically insignificantly different. Thus, the equilibrium model successfully endogenizes the "leverage-effect" as the stock price responds negatively to increases in uncertainty.

This paper computes several measures of the reward to variability (i.e., Sharpe ratios) of volatility strategies. It is shown that the total reward to variability averages to about -0.48 in the theoretical model. This is somewhat lower than reported in the empirical options literature, in which Sharpe ratios for sellers of volatility are reported to range between 0.45 and 1. The model exhibits a large variation in risk premia, implying that investors who sell volatility when premia are high will earn Sharpe ratios that well exceed 0.45. In addition, the paper demonstrates that theoretical options returns and Sharpe ratios earned by investors who sell volatility are relatively high. This means that high Sharpe ratios for options sellers found in previous empirical studies could be consistent with the equilibrium model and its estimated parameters.

The remainder of this paper is organized as follows. The next section presents the model and the theoretical equilibrium framework. Section 3 discusses estimation and data. Section 4 presents the empirical results, including the estimated conditional and unconditional volatility premia, structural estimates of the equilibrium model, the model implied volatility premium, theoretical options returns, as well as various model diagnostics. Section 4 presents out-of-sample results using data through the financial crisis of 2008. Section 5 concludes.

2 Model

The objective of the paper is to present an equilibrium explanation for the volatility premium. To derive a model that even has a chance at generating a significant market price of volatility risk required to explain the premium, one needs to consider non-standard equilibrium constructions. It is not enough to assume, for example, a standard CRRA power utility consumption model because in this model the volatility risk premium will be zero unless volatility correlates directly with consumption. This paper therefore follows Eraker and Shaliastovich (2008) and specifies continuous time long-run risk equilibrium. Unlike Bansal and Yaron (2004), the equilibrium model assumes that consumption growth rates are constant, and that the only channel of variation in expected returns is coming from changes in volatility. It is easy to incorporate additional risk factors such as time-varying expected real consumption growth (as in Bansal and Yaron (2004)), or time-varying inflation risk premia (as in Piazzesi and Schneider (2006), Eraker (2006)). By focusing on volatility as the single driving factor, we avoid having to assess how additional factors impact the premium. The model framework is outlined in detail in Eraker and Shaliastovich (2008) and a brief discussion is given below.

2.1 Assumptions

We consider an endowment economy where a representative agent has Kreps-Porteus-Epstein-Zin recursive preferences,

$$U_t = \left[(1-\delta) C_t^{1-\frac{1}{\psi}} + \delta (E_t U_{t+1}^{1-\gamma})^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{1-\frac{1}{\psi}}. \tag{1}$$

The parameters δ, γ, and ψ represent the subjective discount factor, preference over resolution of uncertainty, and elasticity of substitution, respectively. The KPEZ preference structure collapses to a standard CRRA utility representation if $\gamma = 1/\psi$. It is well understood that the KPEZ preferences lead to the Euler equation

$$E_t \left[\delta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{c,t+1}^{-(1-\theta)} R_{i,t+1} \right] = 1, \tag{2}$$

where $\theta = (1-\gamma)/(1-1/\psi)$, $R_{c,t}$ is the return on aggregate wealth, defined as the present value of future consumption, and $R_{i,t}$ is the return on some arbitrary asset. The dynamics of aggregate wealth are endogenous to the model and depend on the assumed dynamics for consumption. The stock market does not capitalize the entire asset pool in the economy. Rather, it is assumed that the aggregate dividend on market-capitalized assets follows a process, D, which differs from the aggregate consumption process, C. Following Bansal and Yaron (2004) the model allows for time-varying uncertainty in the macro economy, but without time-variation or stochastics in expected growth rates. The model is

$$\begin{aligned} d\ln C &= \mu_c dt + \sqrt{V} dW^c && (3) \\ d\ln D &= \mu_c dt + \phi_d \sqrt{V} dW^c + \sigma_d \sqrt{V} dW^d && (4) \\ dV &= \left[\kappa_v(\bar{v} - V) - l_1 \mu_V \bar{v}\right] dt + \sigma_v \sqrt{V} dW^V + \xi dN && (5) \end{aligned}$$

where dN is a Poisson jump process with arrival intensity proportional to the level of economic uncertainty, $l_1 V$. The volatility process, V, has jump sizes, ξ, assumed to follow a Gamma distribution,

$$\xi \sim GA(\mu_v/r, r)$$

such that $E(\xi) = \mu_v$ and r is the shape parameter.

The model in equations 3 to 5 is a very simple one, and probably offers an overly simplistic view of both the macro-economy as well as asset pricing implications for assets outside the model. For example, the model cannot successfully capture the time-variation in the term structure of interest rates because there is only a single factor, V, which drives expected asset returns. It is easy to generalize the model to allow for additional factors. Since this paper focuses only on the volatility risk premium and the interaction between volatility and stock returns, additional factors are omitted from the model. It should be noted however, that allowing for additional factors such as stochastic growth rates will only increase the premium if shocks to the growth rate (as in Bansal and Yaron) are proportional to the economic uncertainty, V.

2.2 Equilibrium

This paper follows Eraker and Shaliastovich (2008) and derives continuous time equilibrium prices from the KPEZ Euler equation (1). While further details can be found in Eraker and Shaliastovich, the following discussion highlights the essentials.

The price-dividend ratios are given by

$$z_t := \ln P_t - \ln D_t = A_{0,d} + B_{v,d} V_t. \tag{6}$$

where P_t is the time t stock price. This equation illustrates that price-multiples in this economy depend only on the level of economic uncertainty, V. The parameter $B_{v,d}$ determines how stock prices respond to changes in volatility. Since,

$$d \ln P_t = d \ln D_t + B_{v,d} dV_t, \tag{7}$$

the (log) stock price $d\ln P$ responds negatively to changes in volatility whenever $B_{v,d}$ is negative.

There are two priced risk factors in the economy: shocks to consumption dW^c, and shocks to volatility. The latter can come either in terms of "small" shocks dW^V or discontinuous shocks ξdN which can be large. The market price of consumption shocks is simply γ, the "risk aversion," in this model. The market prices of both diffusive and jump volatility shocks are determined by the parameter

$$\lambda_v = (1-\theta)k_1 B_{v,d}. \tag{8}$$

The market price of diffusive volatility shocks is given by

$$\Lambda_t = \lambda_v \sigma_v \sqrt{V_t} \tag{9}$$

and is time-varying since it depends on V_t. Thus, investments in volatility-sensitive assets, such as the aggregate stock market as well as derivatives, command a time-varying risk premium determined by λ_v.

2.3 Equivalent Measure

In the following I discuss the evaluation of derivatives prices and derive an explicit expression for the long-run, unconditional volatility premium. To discuss derivative prices, we use the standard approach in the derivatives literature and specify the dynamics of the economy using an imaginary world adjusted for risk. This risk-neutralized economy is given by

$$
\begin{aligned}
d\ln C &= (\mu_c - \gamma V)dt + \sqrt{V}d\tilde{W}^c && (10)\\
d\ln D &= \mu_c dt + \phi_d\sqrt{V}d\tilde{W}^c + \sigma_d\sqrt{V}dW^d && (11)\\
dV &= \left[\kappa_v(\bar{v} - V) - l_1\mu_V\bar{v} - \lambda_v\sigma_v^2 V\right]dt + \sigma_v\sqrt{V}d\tilde{W}^V + \xi^q dN^q && (12)\\
\xi^Q &\sim GA(\frac{\mu_v}{r + \lambda_v\mu_v}, r) && (13)\\
l_1^Q &= (1 + \frac{\lambda_v\mu_v}{r})^{-r} l_1 && (14)
\end{aligned}
$$

where $\tilde{W}$ denotes Brownian motion under Q, N^q is a Poisson counting process with instantaneous arrival intensity $l_1^Q V_t$, and ξ^Q is the distribution of jump sizes under Q. The parametric restriction $r > \lambda_v\mu_v$ (an implicit restriction on the permissible equilibriums) ensures that the jump intensity and jump size distributions are well-defined. In this case, it is easy to see that jumps arrive *more* frequently and are *greater* in size under the risk neutral measure, whenever $\lambda_v < 0$. This makes it appear as if the risk-neutralized economy has greater and more frequent jumps, and thus market crashes, than what can be objectively inferred from studying the actual economy. This again implies that options prices, which depend directly on the dynamics under the risk neutral measure, reflect risk premia for extreme events that may substantially exceed the frequency and magnitude of the actual events.

The dynamics of the stock price is given by

$$d \ln P_t = \left[(\mu - \phi\gamma V_t) + B_{d,v}(\kappa_v(\bar{v} - V_t) - \lambda_v \sigma_v^2 V_t) - B_{d,v} l_1 \mu_v \bar{v} \right] dt$$
$$+ \sigma_d \sqrt{V_t} dW_{d,t}^Q + \phi \sqrt{V_t} dW_{c,t}^Q + B_{d,v} \sigma_v \sqrt{V_t} dW_{v,t}^Q + B_{d,v} \xi_V^Q dN_t^Q. \quad (15)$$

under the equivalent measure and

$$d \ln P_t = \left[\mu + B_{d,v}(\kappa_v(\bar{v} - V_t)) - B_{d,v} l_1 \mu_v \bar{v} \right] dt$$
$$+ \sigma_d \sqrt{V_t} dW_{d,t}^Q + \phi \sqrt{V_t} dW_{c,t} + B_{d,v} \sigma_v \sqrt{V_t} dW_{v,t} + B_{d,v} \xi_V dN_t. \quad (16)$$

under the objective, observable measure. Note that the volatility shocks enter directly into the dynamics for the stock price with a multiplier equal to $B_{v,d}$. This is true for both the diffusive shocks and the jumps. The correlation between jumps in stock prices and jumps in volatility was found empirically relevant in Eraker, Johannes and Polson (2003). In that model, however, there is no explicit link between the correlation of diffusive shocks in prices and volatility and the correlation in price jumps and volatility jumps. By and large, one of the main advantages of specifying a model using equilibrium arguments is that it takes away the need for guesswork in specifying the stock price as well as the link between the objective and risk-neutral dynamics.

2.4 The Volatility Premium

The volatility premium is defined as the difference between the conditional variance (or standard deviation) of the (log) stock price some τ periods ahead,

$$VP_t(\tau) = \mathrm{Var}_t^Q(\ln P_{t+\tau}) - \mathrm{Var}_t(\ln P_{t+\tau}). \tag{17}$$

We can compute the premium from knowledge of the moment generating function $\Phi_i(u,\tau) = E_t^i \exp(u \ln P_{t+\tau})$ for $i = \{P, Q\}$. The volatility premium is computed using the fact that $\mathrm{Var}_t^i(\ln P_{t+\tau}) = \partial^2 \ln \Phi_i(u)/\partial u^2|_{u=0}$ which is found numerically by solving the the standard ODE's that give the generating functions for the (log) stock price. Since the generating function is of the affine form $\Phi_i(u,\tau) = \exp(\alpha_i(u,\tau) + \beta_i(u,\tau)V_t)$ we have that the

$$\mathrm{Var}_t^i(\ln P_{t+\tau}) = \alpha_i''(0,\tau) + \beta_i''(0,\tau)V_t. \tag{18}$$

In particular, the (squared) VIX index is the conditional variance 22 days (one month) ahead is a linear function of the underlying macro-volatility, V_t,

$$VIX_t^2 = \mathrm{Var}_t^Q(\ln P_{t+22}) = \alpha_Q''(0,22) + \beta_Q''(0,22)V_t \tag{19}$$

$$=: \alpha_v + \beta_v V_t. \tag{20}$$

This equation is used for econometric identification as will become clear below. While we compute the volatility premium numerically at one-month horizons corresponding to the theoretical computation of the VIX index, it is easy to see from (15) and (16) that the premium is zero as $\tau \to 0$ if the model does not have volatility jumps. A volatility process with continuous paths and no jumps carries an unconditional, long-run volatility premium.

3 Estimation and Data

In order to construct inference for the volatility premium, this paper employs full structural likelihood-based inference for the underlying equilibrium model. This is carried out by formulating a likelihood function which relies on the equilibrium dynamics of stock prices and volatility. The equilibrium solution is characterized by the parameters $\{k_1, B_v, k_{1,d}, B_{d,v}, \lambda_v, A_0, A_{0,d}, \alpha_v, \beta_v\}$. Since these parameters are non-linear functions of the structural parameters in the model, we need to solve for these equilibrium parameters for each iteration of the likelihood function. This poses a significant numerical estimation challenge. The following discussion gives an overview of the estimation approach.

Let $Y_t = (\tilde{R}_t, VIX_t^2)$ denote the observed returns $\tilde{R}_t = \int_{t-1}^{t} d\ln R_s$ and implied volatility data. Let $X_t = (\ln D_t, V_t)$ be the unobserved dividend and macro-volatility processes. We have that

$$\begin{aligned}\begin{bmatrix} \tilde{R}_t \\ VIX_t^2 \end{bmatrix} &= \begin{bmatrix} k_0 + (k_1 - 1)A_{0,d} \\ \alpha_v \end{bmatrix} \\ &+ \begin{bmatrix} 1 & k_d B_{v,d} \\ \beta_v & 0 \end{bmatrix} \begin{bmatrix} \ln D_t \\ V_t \end{bmatrix} - \begin{bmatrix} 1 & B_d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ln D_{t-1} \\ V_{t-1} \end{bmatrix},\end{aligned} \tag{21}$$

or more compactly

$$Y_t = \alpha + \bar{\beta} X_t + \bar{\bar{\beta}} X_{t-1}.$$

Since there is a one-to-one map between the unobserved state-variables $X_t = (lnD_t, V_t)$ and the observed data Y_t, we can solve for the states given a parameter Θ^*. Define

$$X_t^* = \{X_t \mid Y_t = \alpha^* + \bar{\beta}^* X_t + \bar{\bar{\beta}}^* X_{t-1}\}$$

where $\alpha^*, \bar{\beta}^*$ and $\bar{\bar{\beta}}^*$ are equilibrium solutions at Θ^*. The likelihood function can now be computed from

$$\ln \mathcal{L} = \sum_{t=1}^{T} \ln p_x(X_t^* \mid X_{t-1}^*, \Theta^*) - \frac{T}{2} \ln |\bar{\beta}\bar{\beta}'| \tag{22}$$

where $p_x(X_t \mid X_{t-1}, \Theta)$ is the transition density of the jump-diffusion process X_t.

Several methods can be used to compute $p_x(. \mid .)$. This paper relies on a simulation based estimator which involves sampling the jump times $\triangle J_{t_i}$ as well as jump sizes ξ_{t_i} and artificial sampling intervals $t_i = t + i\triangle$ for $\triangle = 1/m$ where m is chosen by the econometrician. This approach follows Eraker (2001) and Eraker, Johannes, and Polson (2003) among others. The availability of data on the VIX index limits the sample size to 1990-on. This paper uses end of day data for S&P 500 log-returns as well as VIX data from 1990 until the end of 2006. This yields a total of 4286 daily observations.

4 Empirical Results

4.1 Descriptive Evidence

Figure 1 presents exploratory evidence on the behavior of the VIX volatility index, as well as the volatility premium. In order to gauge the volatility premium, I constructed a model-based forecast of the 22 day ahead integrated variance

$$\mathrm{Var}_t^P(\ln P_{t+22}) = E_t \int_t^{t+22} \sigma_s^2 ds$$

which amounts to the theoretical variance of the stock returns under the "observable" measure P. The square-root of this quantity is referred to as the conditional standard deviation in the figure. The difference between the VIX index and the model-based forecasted conditional standard deviation is a noisy estimate of the conditional volatility premium. As

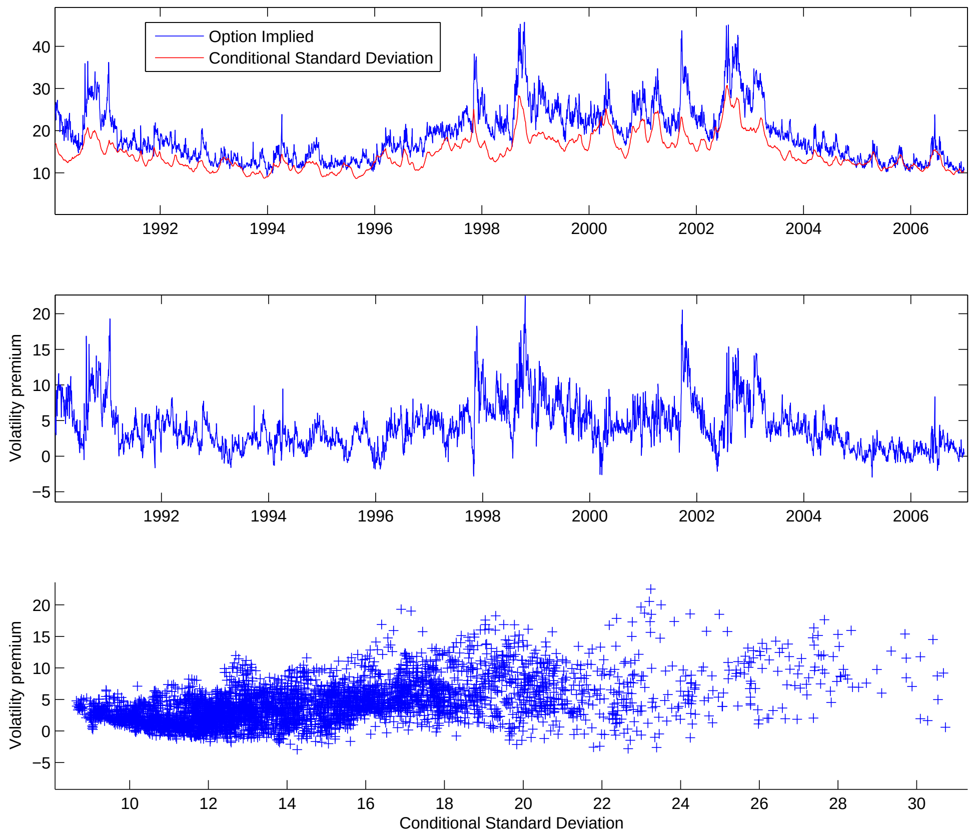

Figure 1: Option implied (VIX) data and conditional standard deviation. Top: VIX and conditional standard deviation. Middle: The volatility premium (in units of standard deviation). Bottom: Scatter plot of option implied and conditional standard deviation.

can be seen in the upper plot in Figure 1, the VIX index generally exceeds the P forecast, giving a positive difference shown in the middle graph. The evidence is broadly consistent with the exploratory evidence in Todorov (2007) who uses a much more elaborate model to forecast the integrated variance.

In the equilibrium model the market price of volatility risk increases proportionally to the level of the conditional variance. The volatility premium is also increasing in the

Table 1: Unconditional Volatility Premium

The table presents estimates of the volatility premium in standard deviation and variance units. The premium in standard deviation units is defined as $\hat{E}(VIX) - \text{Std}(\tilde{R}_t)\sqrt{252}$ and in variance units $\hat{E}(VIX^2) - \text{Var}(\tilde{R}_t)252$ where $\tilde{R}_t$ are daily returns on S&P 500. Percentiles of the sampling distributions are computed by block bootstrap using one year blocks.

			Percentiles						
	mean	std	1%	5%	10%	50%	90%	95%	99%
Std units	0.033	0.0044	0.022	0.025	0.027	0.033	0.038	0.04	0.042
Var units	0.015	0.0016	0.012	0.013	0.013	0.015	0.017	0.018	0.019

conditional variance. The bottom plot in Figure 1 is a scatter plot of the level of volatility and the volatility premium (standard deviation units). The plot suggests that the premium is increasing on average with the level of volatility, and the correlation between the two is about 0.4. While the correlation is far from perfect, this crude evidence does indeed suggest that the premium on average increases when volatility is high, consistent with the equilibrium model. This is also consistent with the empirical evidence in Bakshi and Kapadia (2003). They show that delta-hedged gains from writing options increase with the level of volatility from about 1.7 % of the initial price when annualized volatility is less than 8%, to more than 22 % when volatility exceeds 18%.

Table 1 assesses the unconditional premium, defined as the difference between the mean variance and standard deviation implied by the VIX index and simple, annualized estimates of the unconditional stock return variance and standard deviation. The table reveals that the premium is substantial: It amounts to 3.3 percentage points per annum in standard deviation units, and 1.5 in variance units. Irrespective of units of measurement, the volatility premium is significantly positive, as indicated by the lower percentiles of the sampling distribution given in the rightmost columns of the table. For example, the lower one-percentile of the sampling distribution is 2.2 annualized percentage points. This suggests that the premium is economically and statistically significant.

4.2 Estimates of Structural Parameters

Table 2 gives estimates of the structural parameters of the model. First off, the preference over uncertainty resolution, γ, is 15.8. This is higher than the values calibrated to give the appropriate equity premium and equity volatility by Bansal and Yaron (2004). It is almost identical to Bansal, Kiku, and Yaron (2006) where γ is estimated to be 15.12 in the BY model with stochastic volatility. Bansal, Gallant, and Tauchen (2007) estimate γ to be 7.14 with ψ constrained to 2.

There is considerable debate in the literature over what is the "true" value of intertemporal elasticity of substitution, ψ. Most of the literature, including Hansen and Singleton (1982), Vissing-Jørgensen (2002), Guvenen (2005), produces somewhat conflicting evidence. The literature is largely working from an identifying assumption that IES can be found through an instrumental variables regression of consumption growth onto interest rates. This estimating equation is derived under CRRA utility. It *does* not apply in the context of long-run risk models based on KPEZ utility, and it is straightforward to show that if a

Table 2: Parameter Estimates - Two Factor Model

The table reports posterior means of the preference parameters γ (coefficient determining the timing resolution of uncertainty), ψ (elasticity of substitution), and parameters determining the evolution of exogenous state dynamics for consumption, dividends and consumption volatility, given by

$$\begin{aligned} d\ln C &= \mu_c dt + \sqrt{V} dW^c, \\ d\ln D &= \mu_c dt + \phi_d \sqrt{V} dW^c + \sigma_d \sqrt{V} dW^d, \\ dV &= \kappa_v(\bar{v}(1 - l_1\mu_v) - V)dt + \sqrt{V} dW^v + \xi dN. \end{aligned}$$

Long-run average consumption growth is fixed at 0.02 per annum while average consumption volatility, $\bar{v}$, is fixed at $0.03^2/252$ corresponding to an annual consumption growth rate standard deviation of 0.03. Results shown are the posterior means and standard deviations of model parameters based on daily data on S&P 500 and the VIX index from 1990-2006 (4286 obs.).

Preference and Risk Parameters

	γ	ψ	$(\delta-1)*100$	λ_v	l_1^Q	μ_v^q
Posterior Mean	15.8	1.48	0.0185	-42,615	183	1.25e-5
Posterior Std.	(0.175)	(0.0625)	(0.000419)	(653.96)	(1.80)	(6.4e-7)

System Parameters

	κ_v	σ_v	σ_d	ϕ_d	l_1	μ_v	r
Posterior Mean	0.00474	0.00019	1.99	3.99	118.6	8.33e-6	1.01
Posterior Std.	(3.11e-5)	(1.44e-6)	(0.129)	(0.118)	(1.6)	(7.79e-8)	(0.0173)

long-run risk model generates the data, the IV regression approach produces biased estimates of IES. Table 2 reports an estimate of $\psi = 1.48$, while in disagreement with most of the estimates of IES produced based on CRRA preferences, is not unreasonable. To see why, note that wealth, defined as the present value of consumption, would actually increase as a function of volatility in this model if $\psi < 1$.[2] Bansal & Yaron (2004) and Eraker (2007a) show that values of ψ less than unity produce too high interest rates and too low equity premia.

Table 2 also gives estimates of the parameters that describe the evolution of the exogenous dividend and volatility processes. Perhaps the most interesting parameter here is the speed of mean reversion for the volatility process, κ_v. This is estimated to be 0.00474 which corresponds to a daily autocorrelation of volatility of about 0.9963[3]. This implies a very persistent process, and the amount of persistence somewhat exceeds those typically found in the time-series literature[4]. My estimate of the volatility persistence implies a half-life of volatility shocks of about six and a half months. It is well documented that estimates of volatility persistence increase as sampling frequency is made coarser[5]. The reason why the persistence is found somewhat higher than that typically found in daily data is that the structural model implies a very close tie between the volatility persistence and the size of the volatility risk premium. In this model, the volatility premium increases uniformly as κ_v

[2]This is a standard result in LRR models. See for example Bansal & Yaron (2004), eqn. (A7).

[3]This estimate obtains as $\exp(-\kappa_v + \mu_v l_1)$.

[4]Typical autocorrelation estimates range from 0.97 to above 0.99. For example, Eraker, Johannes and Polson (2003) find κ_v ranging from about 0.0128 to 0.026. GARCH(1,1) estimates obtained here for the S&P 500 returns imply an AR(1) coefficient of 0.9915.

[5]Chacko and Viciera (2005) find volatility half-life ranging of 2 and 16 years using monthly and annual returns data, respectively.

decreases. As such, evidence in the data of a high premium is consistent with long-range dependence.

The estimates of the jump parameters in the model are suggestive of extremely rare, but large volatility jumps. The jump intensity in the model is proportional to the level of the volatility process, $l_1 V_t$. The estimate of $l_1 = 118.6$ implies that jumps occur on average every 10th year. When they do occur, the average jump size is more than twice that of the long-run average volatility. Under the risk neutral measure, jumps occur much more frequently with an estimated arrival intensity of $183V_t$ corresponding to a jump every sixth year, or about 50% more frequently than under the objective measure. Jump sizes are also about 50% greater under the risk neutral distribution. These risk adjustments potentially lead to sizable premia for jump risks in options markets.

4.3 Other Asset Price Implications

Table 3: Asset Price Implications

The table examines key moments of observed and model implied asset market data. The p-value is a model based bootstrap giving the probability of observing a sample path with the same moment as computed in the data.

	data	model	p-value
Equity Premium	5.9	6.9	0.62
return std	0.99	1.10	0.36
Corr($\triangle V, r$)	-0.72	-0.66	0.16

Table 3 computes key asset price characteristics generated by the equilibrium model. These numbers were computed by simulating returns and volatility data using the parameter estimates in Table 2. The table reveals that the equity premium generated by the model is 6.9 %, which compares to 5.9% in the data over the 1990-2006 sample period.[6] The 6.9% equity premium is close to the average excess return of about 7.5% computed for a longer sample period in the US market. The equilibrium model produces a population standard deviation of stock returns which slightly exceeds the sample standard deviation. The equilibrium model produces a correlation between changes in volatility and changes in stock prices averaging to -0.66. This is somewhat lower than in the data for which the correlation is -0.72 over the sample period. This difference is not statistically significant. Neither are the differences between any of the other model implied moments and the observed data. As such, one cannot reject the null-hypothesis that the model is in fact the true data-generating process by looking at these moment-based tests by themselves. While slightly lower than in the data, the -0.66 correlation between volatility is really a substantial feat. The correlation is entirely an endogenous equilibrium effect where stock prices fall in response to increases in uncertainty. I am unaware of any other equilibrium model that comes even close in substantiating the whole asymmetric volatility relationship.

Table 4: Volatility Premium. In-Sample Evidence

The table computes the posterior means and standard deviation of the two measures of the unconditional premium,

$$VP = \alpha_v^q + \beta_v^q \hat{E}\hat{V}_t - \alpha_v^p - \beta_v^p \hat{E}\hat{V}_t$$

(variance units) and

$$SP = \hat{E}\sqrt{\alpha_v^q + \beta_v^q \hat{V}_t} - \hat{E}\sqrt{\alpha_v^p - \beta_v^p \hat{V}_t}$$

(standard deviation units) for the in-sample extracted estimate $\hat{V}_t$ of macro-volatility. $\hat{E}$ is the sample mean, $\frac{1}{T}\sum_t$.

	VP		SP
Data	0.015		0.033
		Model	
Posterior Mean	0.014		0.038
Posterior Std.	0.001		0.003

4.4 The Volatility Premium

Turning to the focal point of this paper, table 4 reports estimates of the unconditional, average volatility premium based on the in-sample parameter estimates and the estimated volatility path. The table shows that the volatility premium is estimated to be fairly close to that observed in the data. Using variance units, the model produces a variance premium of 0.014 (posterior mean) which compares to 0.015 in the data. In units of standard-deviation, the model produces 0.038 which surprisingly exceeds the unconditional number of 0.033 computed in the data. This may be due to the fact that the number computed from the data is based on a different estimator than the theoretical, model implied numbers given in table 4. In constructing a frequentist test of statistical significance of the difference in the computed premium, we can compare the model implied premium in table 4 to the percentiles of the sampling distribution in table 1. Using variance units, we find that the model-implied 0.014 lies above the lower 10th percentile, rendering the difference insignificant by a one-sided test. Similarly, the 0.038 model-implied difference in the standard deviations fall right on the 90th percentile of the sampling distribution in table 1. Neither measure of the premium, therefore, can be concluded to be statistically different from the one observed in the data at high levels of confidence.

[6] The 5.9% refers to the total return on the S&P 500 index, as measured by the S&P 500 total return index which, unlike the widely quoted S&P 500 index (SPX), includes dividends. The average return (capital gains only) on the S& P 500 index is about 3.7% above the risk free rate per year.

4.5 Risk Premia

It is common in the no-arbitrage literature to specify the exogenous processes driving market prices of risk to allow various risks. Market prices of risk have the interpretation of being the expected instantaneous reward per unit of standard deviation, or a continuously computed Sharpe-ratio. In the equilibrium framework, market prices of risks are generated endogenously from the preferences and the parameters that determine the dynamic behavior. The annualized market price of risk for consumption in the equilibrium model is given by $\gamma\sqrt{V}\sqrt{252}$ which is about 0.47 on average in my model.

It is of course particularly interesting to compute the reward to volatility risk. This implicitly will determine whether the model can explain seemingly high Sharpe ratios to issuers of equity options. It is straightforward and sensible to compute the market-price-of-risk for the locally normally distributed shocks to the volatility process. It is $\lambda_v \sigma_v \sqrt{V}$. For jumps, accordingly $[E(\xi dN) - E^Q(\xi dN)]/\mathrm{Std}_t(\xi dN)$ is the reward to a hypothetical investment in the jump part of the volatility process.[7]

Figure 2 plots the reward-to-risks for the diffusive volatility part, the jump part, and the total risks. The latter can be interpreted as the instantaneous Sharpe-ratio earned by an investor who invests directly in volatility, either by buying options or volatility futures contracts. The premium for diffusive risks fluctuates between -0.09 to -0.49, the premium

[7] Since the reward-to-risk is defined in terms of the first two moments it does not adequately reflect the risk-return tradeoff for investors who have more general preferences than mean-variance utility since the jump sizes are non-normally distributed.

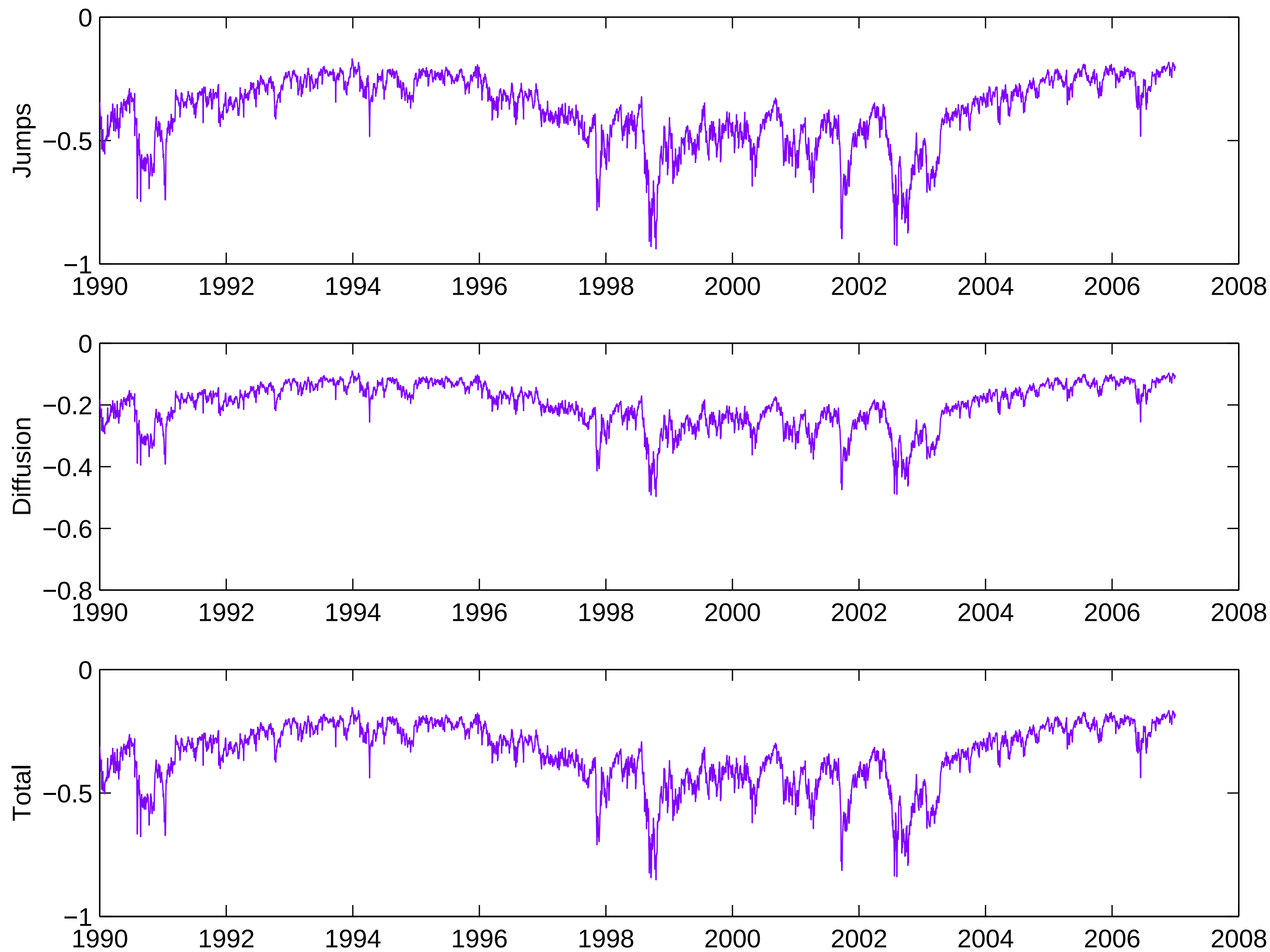

Figure 2: Top:Reward to variability for volatility jumps. Middle: Market price of diffusive volatility risk. Bottom: The total reward to volatility (diffusive+jump) risk.

for jump risk fluctuates between -0.16 and -0.94, and the total premium fluctuates between -0.15 and -0.85. It is reasonable that the jump risks carry a higher premium because of the non-normality of jumps in the volatility process. The average annual total reward-to-variability is only about -0.35. This is surprisingly small, particularly in light of the empirical evidence in the options literature that the investors who sell volatility earn Sharpe ratios between one half and one. There are three possible explanations to this. First, it could be that the empirical evidence cited is based on returns over a period in which the rewards-to-variability was higher than the sample period used here. Second, it could be that the returns to sellers of options are upwardly biased if the volatility went down on average over the sampling period because if volatility goes down, a short volatility position essentially produces a return equal to the volatility premium plus returns generated by the negative of the directional move in the volatility process. Third, the options return literature uses data that typically exclude extreme market events such as the crash of '87 and the '08 financial crisis leading to an inherent peso problem in the sample selection.

4.6 Option Returns

Are high average returns on short option positions reported in the literature consistent with equilibrium? I present two pieces of evidence to shed light on this. First, Figure 3 plots implied Black-Scholes volatility for one month options computed using the equilibrium model.[8] The figure illustrates that the implied volatility computed from the model is largely consistent with those observed empirically. First, there is a pronounced smile. Second, the highest implied volatilities obtain for the low strikes. This is consistent with the well known fact that out-of-the-money put options are very expensive.

Table 5 reports simulated returns and Sharpe-ratios from one-month long investments in options under different volatility scenarios. The purpose is to see if the return patterns implied by the model are consistent with return and risk patterns suggested in the empirical

[8]The implied volatilities were computed by equating the Black-Scholes model price to the theoretical equilibrium price using the equilibrium interest rate and dividend yield. Details on how to compute the equilibrium options prices can be found in Eraker & Shaliastovich (2008).

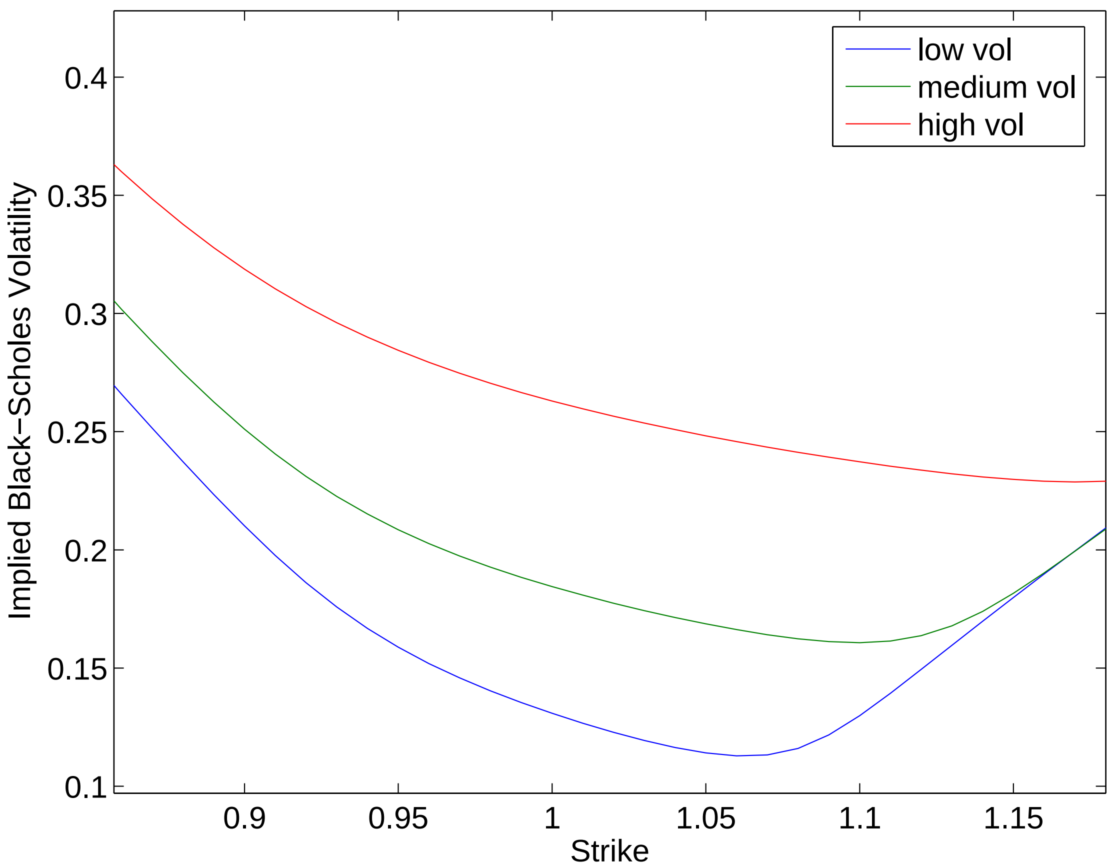

Figure 3: Implied Black-Scholes volatility for a one-month option under different initial volatility regimes.

Table 5: Simulated Option Returns

The table reports simulated returns and Sharpe ratios on option positions. The simulations assume that the option is traded at the theoretical price, C_t, computed from the equilibrium model using estimated parameters in table 2. Returns are arithmetic returns assuming the option is held until expiration.

	mean returns			Sharpe ratios		
Strike	Calls	Puts	Straddles	Calls	Puts	Straddles
			High initial volatility			
0.85	0.08	-0.73	0.06	0.16	-0.20	0.13
0.9	0.10	-0.60	0.05	0.15	-0.22	0.09
0.95	0.12	-0.43	0.01	0.12	-0.24	0.01
1	0.13	-0.29	-0.09	0.09	-0.23	-0.13
1.05	0.12	-0.19	-0.14	0.05	-0.21	-0.21
1.1	0.07	-0.12	-0.12	0.01	-0.20	-0.20
1.15	-0.08	-0.08	-0.08	-0.01	-0.19	-0.19
			Medium initial volatility			
0.85	0.03	-0.71	0.025	0.09	-0.13	0.07
0.9	0.04	-0.61	0.024	0.08	-0.13	0.05
0.95	0.05	-0.39	-0.004	0.06	-0.15	-0.01
1	0.03	-0.20	-0.092	0.02	-0.15	-0.13
1.05	-0.05	-0.09	-0.093	-0.02	-0.12	-0.14
1.1	-0.34	-0.05	-0.045	-0.05	-0.11	-0.11
1.15	-0.94	-0.03	-0.032	-0.23	-0.10	-0.11
			Low initial volatility			
0.85	0.01	-0.77	0.00	0.01	-0.10	-0.00
0.9	0.01	-0.69	-0.00	0.01	-0.11	-0.02
0.95	-0.00	-0.41	-0.03	-0.01	-0.12	-0.06
1	-0.07	-0.13	-0.10	-0.06	-0.09	-0.14
1.05	-0.32	-0.02	-0.03	-0.09	-0.04	-0.05
1.1	-0.94	-0.01	-0.01	-0.30	-0.03	-0.03
1.15	-1	-0.00	-0.00	-Inf	-0.03	-0.03

Table 6: Simulated Delta-Hedged Option Returns

The table reports simulated returns and Sharpe ratios on option positions and short stock positions. The positions are short delta number of futures contracts on the stock at initiation.

	mean returns			Sharpe ratios		
strike	Calls	Puts	Straddles	Calls	Puts	Straddles
			High initial volatility			
0.85	-0.01	-0.61	-0.02	-0.16	-0.17	-0.17
0.9	-0.02	-0.43	-0.05	-0.16	-0.17	-0.17
0.95	-0.05	-0.23	-0.09	-0.16	-0.17	-0.16
1	-0.11	-0.11	-0.11	-0.15	-0.15	-0.15
1.05	-0.23	-0.04	-0.07	-0.13	-0.14	-0.14
1.1	-0.43	-0.01	-0.02	-0.11	-0.12	-0.11
1.15	-0.86	-0.00	-0.00	-0.09	-0.13	-0.11
			Medium initial volatility			
0.85	-0.00	-0.71	-0.01	-0.11	-0.13	-0.12
0.9	-0.01	-0.56	-0.02	-0.12	-0.13	-0.13
0.95	-0.03	-0.29	-0.06	-0.14	-0.15	-0.14
1	-0.11	-0.10	-0.10	-0.14	-0.14	-0.14
1.05	-0.29	-0.02	-0.03	-0.12	-0.13	-0.12
1.1	-0.69	0.00	-0.00	-0.10	-0.13	-0.11
1.15	-1.40	0.00	0.00	-0.22	-0.38	-0.30
			Low initial volatility			
0.85	0.00	-0.68	-0.00	-0.06	-0.08	-0.07
0.9	-0.00	-0.59	-0.01	-0.07	-0.08	-0.08
0.95	-0.02	-0.34	-0.04	-0.10	-0.10	-0.10
1	-0.10	-0.09	-0.10	-0.12	-0.13	-0.12
1.05	-0.40	-0.00	-0.01	-0.10	-0.12	-0.11
1.1	-1.00	0.00	0.00	-0.22	-0.33	-0.28
1.15	-1.10	0.00	0.00	-0.32	-0.45	-0.39

options returns literature. I distinguish between low, medium, and high initial volatility, V_t.

Put options lose money on average irrespective of the initial volatility state. Far out-of-the-money puts (strike=0.85) lose between 71 and 77 percent of their value if held until expiration. This illustrates that the risk premium imbedded in prices of out-of-the-money puts comprise the largest component of the price. Driessen and Maenhout (2006) report weekly excess returns for at-the-money puts to be averaging -6% which corresponds to our weekly return of -20%/4=-5% for the average volatility regime in the equilibrium model. Furthermore, they report weekly returns on 4% and 6% out-of-the-money puts to be -7.6 and -8.6%. This compares to weekly returns of about -10% in the equilibrium model. Thus, the empirical evidence in Driessen and Maenhout (2006) matches the theoretical returns in our model quite closely.

Call options have a more complicated return pattern. As seen int Table 5, in-the-money (ITM) calls tend to yield positive returns while out-of-the-money calls yield negative returns. The distinction between an ITM and OTM call is that the former has a higher delta, representing an investment with high directional stock price risk. For ITM calls, the positive beta leads to a risk premium that swamps the negative premium stemming from the volatility exposure. The OTM call, conversely, has less directional price exposure and most of its premium is negative risk stemming from the volatility premium. The results are consistent with tabulated returns for call options reported in Driessen and Maenhout

(2006) who find that out-of-the-money calls have higher positive returns than at-the-money calls.

Table 5 also reports monthly simulated Sharpe ratios for the options, as well as simulated returns and Sharpe ratios on straddle positions. For at-the-money straddles, monthly Sharpe ratios are about -0.14 irrespective of volatility regimes, corresponding to annualized Sharpe ratios of about -0.49. This is about half of the annualized Sharpe ratios for crash protected straddles found in Coval and Shumway (2001), and also somewhat lower than Sharpe ratios for at-the-money straddles reported by Driessen and Maenhout (2006) which can be inferred to be about -0.72. It is almost identical to the Sharpe ratio reported in Eraker (2007b) of 0.46 for an investor who sells options at the market bid price. Notice that investors who sell straddles with strikes slightly higher than the initial stock price ($1), will earn Sharpe ratios of about 0.2, or about 0.69 annualized when the initial volatility is high.

Table 6 offers a slightly different perspective on the returns available from selling options. This table considers returns on options positions where the investor, at the same time as buying the options, simultaneously sells delta number of forward contracts on the stock.[9] In economies such as the theoretical equilibrium economy considered here, where jumps and stochastic volatility affect the stock and options prices, delta-hedging does not provide a perfect hedge but may still eliminate some of the directional price exposure in options positions.

[9] Delta is the partial derivative of the theoretical options price with respect to the initial stock price. In the seminal Black-Scholes analysis, a continuously delta-hedged options position perfectly replicates the payoff on the option. Here deltas are computed using the theoretical model price.

The delta-hedged portfolio returns in Table 6 provide some interesting comparisons to the non-hedged ones in table 5. For example, while returns to calls in the high and medium volatility regimes are positive in table 5, they are negative in table 6. This suggests that the reason why buying call options is profitable is simply that they provide a positive exposure to stock price or market risk. Thus for example, while buying a naked 10% in-the-money call yields returns of 10%,4% and 1 % across volatilities, one obtains -2%,-1% and 0% returns when simultaneously selling delta (close to one in this case) shares of the underlying.

An interesting fact of table 6 is that the average returns and thus the corresponding Sharpe ratios are uniformly negative. Sharpe ratios for at-the-money straddles are between -12 and -14 percent (-42 to - 52% annualized). This is close to what was reported in 5 because at-the-money straddles are approximately market neutral so that the delta-position is close to zero.

So how do the simulated options returns compare to the returns seen in real options data? Table 7 reports average holding period returns for 30 day buy-and-hold positions in S&P 500 index options. These results are comparable to those in Table 5 for the model simulations. As seen, the numbers in Table 7 are quite similar to the simulated numbers in Table 5. In particular, the estimated average returns to put options are quite similar in the model simulations as in the observed data. For example, 85-90% out of the money puts have average returns of -70% per month in our sample whereas the simulated data implies average returns in the -60 to 69% range for the various initial volatility scenarios.

Table 7: S&P 500 Option Returns

The table reports returns and Sharpe ratios on option positions using S&P 500 options data. Returns are computed from buy and hold positions with thirty calendar days until expiration using daily closing prices from September 1996 to May of 2011.

	Mean returns			Sharpe Ratios		
Strike	Calls	Puts	Straddles	Calls	Puts	Straddles
			Average Returns			
0.85	-0.03	-0.78	-0.03	-0.10	-0.39	-0.12
0.90	-0.01	-0.70	-0.02	-0.04	-0.33	-0.07
0.95	-0.03	-0.50	-0.06	-0.06	-0.28	-0.12
1.00	-0.04	-0.11	-0.07	-0.04	-0.08	-0.10
1.05	-0.36	0.03	-0.00	-0.17	0.02	-0.01
1.10	-0.77	0.06	0.05	-0.32	0.09	0.07
1.15	-1.00	0.05	0.04	N/A	0.10	0.08
	# observations					
0.85	1348	1351	1348			
0.90	1803	1809	1803			
0.95	2288	2289	2288			
1.00	2711	2711	2711			
1.05	2188	2177	2177			
1.10	1121	1121	1121			
1.15	698	698	698			

At-the-money straddle returns averaged -7% in the data and between -9 and -10% in the simulations. Overall, the return patterns are qualitatively similar in the data and in the model.

5 Out-of-Sample during the 2008 Financial Crisis

The 2008 financial crisis constitutes the most dramatic financial market event since the Great Depression. While significant Wall Street companies failed, stock market volatility reached extreme levels and stock prices dropped significantly. All the results of this paper reported so far were computed prior to these events. The 2008 financial crisis therefore serves as the ultimate out-of-sample test for our model.

The asset pricing model considered here makes two basic predictions: first, it suggests that the volatility premium is positive and increasing in the level of spot volatility. Thus, we expect to see the premium increase as market volatility virtually exploded in the Fall of 2008. Second, it predicts that a large increase in volatility, ether driven by sudden jumps or by a sequence of smaller (Brownian) shocks, should be associated with a large negative stock price reaction. I examine these predictions in turn.

Table 8: Volatility Premium. Out-of-Sample Evidence

The table reports realized and option implied volatility estimates for the out sample period Jan 2006 - Sept. 2009. $\mathrm{Var}^i(\tilde{R})$ and $\mathrm{Std}^i(\tilde{R})$ for $i = P$ are annualized estimates of integrated variance/ standard deviation and $i = Q$ is the option implied counterpart. The premium is defined $\mathrm{Var}^Q(\tilde{R}) - \mathrm{Var}^P(\tilde{R})$ where $\tilde{R}$ is one month log-return. The t-statistics test the null hypothesis that the differences between the model and data are zero and are computed using Newey-West standard errors.

		Variance Units		
$\mathrm{Var}^P(\tilde{R})$	$\mathrm{Var}^Q(\tilde{R})$	premium (data)	premium (model)	t-stat
0.0914	0.0958	0.00444	0.0369	-1.66
		Std. dev. Units		
$\mathrm{Std}^P(\tilde{R})$	$\mathrm{Std}^Q(\tilde{R})$	premium (data)	premium (model)	t-stat
0.259	0.277	0.0177	0.0468	-1.84

5.1 Volatility Premium Out-of-Sample

In order to evaluate the model's prediction for stock prices throughout the out-of-sample period, consider the following basic facts: At the beginning of 2007, the VIX index was near its historic low, hitting 10% on several days during the first two months of the year. The S&P 500 was near its all time high during 2007January, and hit its all time-high of 1565 two months later. Towards the end of August 2009, the VIX index hovered around 19-20% - almost exactly its historical average. Less than two months later, it hit 80.

Table 8 presents out-of-sample evidence on the volatility premium. Since the model predicts that the premium should increase proportionally to the level of spot volatility, the sharp increase in volatility observed in the out-of-sample period naturally leads to a higher model induced premium. In terms of standard deviation, the model suggests that the premium should have been 4.68% on average over the out-of-sample period. This exceeds the 3.3% reported for the in-sample period. Yet, the data suggest that the premium was only 1.77% on average, implying that the premium decreased during the crisis. To see how the premium presumably was decreased during the crisis, consider figure 4 which depicts the conditional annualized variance premium. The premium is defined as $12(\text{Var}_t^Q(r_{t:t+22}) - \text{Var}_t^P(r_{t:t+22}))$ (dots), along with the model implied premium (solid line). Panel A in this figure clearly shows how the "data" fall below the model predicted line as the spot volatility (on the horizontal axis) increases.

To understand why the model may over-predict the premium throughout the crisis, consider the last term,

$$\text{Var}_t^P(r_{t:T}) = \int_t^T E_t V_s ds = \frac{1}{\kappa_v}\left[(1 - e^{\kappa T - t})(V_t - \theta) + \theta(T - t)\right]$$

where $\theta = E(V)$ is the unconditional variance. This expression shows that the conditional variance depends heavily on the value of κ and it is easy to see that as κ increases, the conditional variance $\text{Var}_t^P(r_{t:T})$ approaches $E(V)(T - t)$. Panel B in figure 4 shows what happens to the premium if we inflate κ from its estimated value of 0.00474 to 0.09. As seen in figure 4 B, the "data" line up almost perfectly with the model. The economic interpretation of this is the following: The crisis led to extremely high levels of volatility, but market participants expected these volatility levels to persist for a relatively short time.

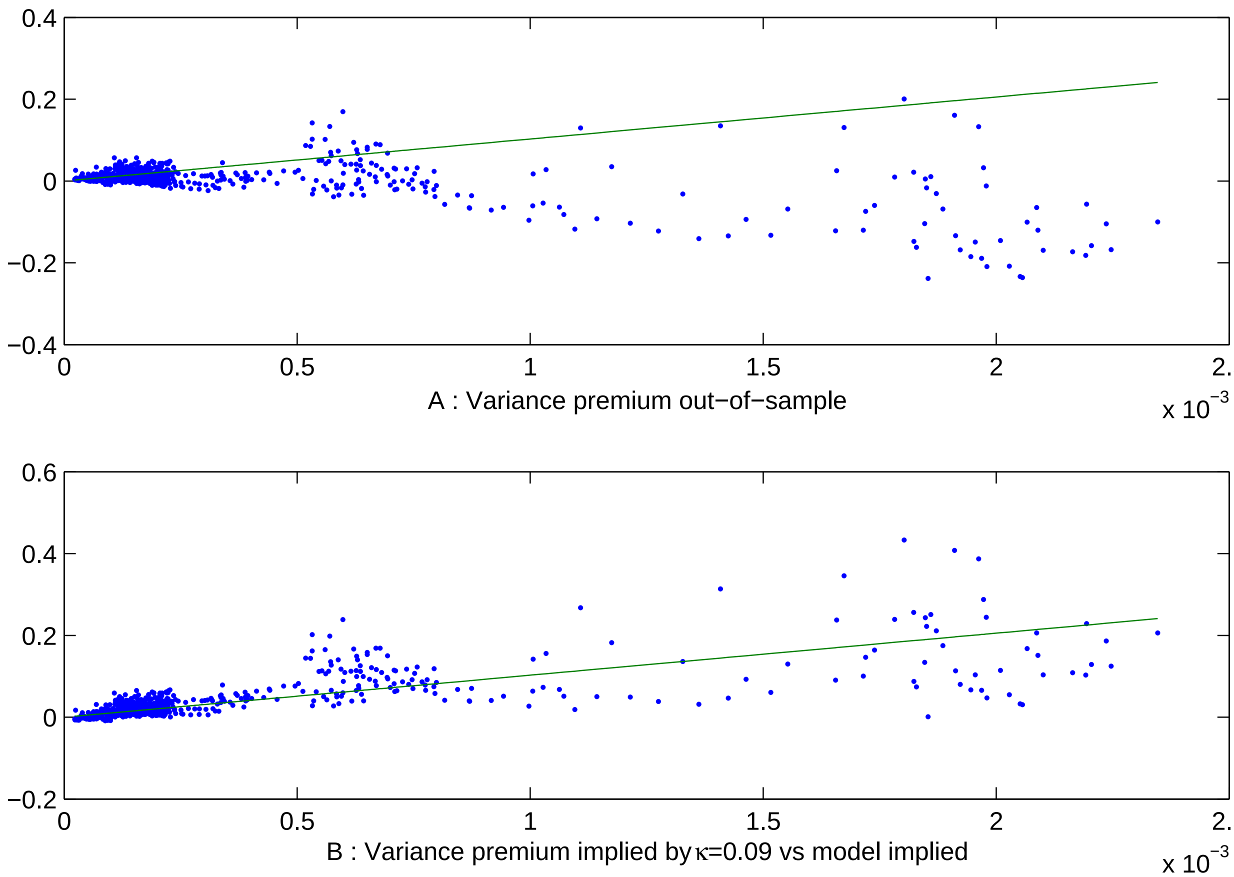

Figure 4: Out-of-sample variance premium.

What the data seem to suggest therefore, is that during high volatility regimes, volatility reverts faster to its long-run mean. In other words, this particular piece of evidence suggests that mean-reversion in financial market volatility is *non-linear* in the level of volatility. Further research on the topic of volatility risk premia should seek to generalize the dynamic specification for the volatility process and go beyond the affine class.[10]

[10]The value of 0.09 was calibrated rather than estimated. I am unaware of statistical evidence suggesting that volatility reverts that quickly and non-linearly from high levels. Some papers, including Bates (2000) and Chernov, Gallant, Ghysels, and Tauchen (2003) however have suggested that volatility should be modeled by two separate factors, one of which reverts quickly. These models are in principle consistent with implied rapid mean reversion.

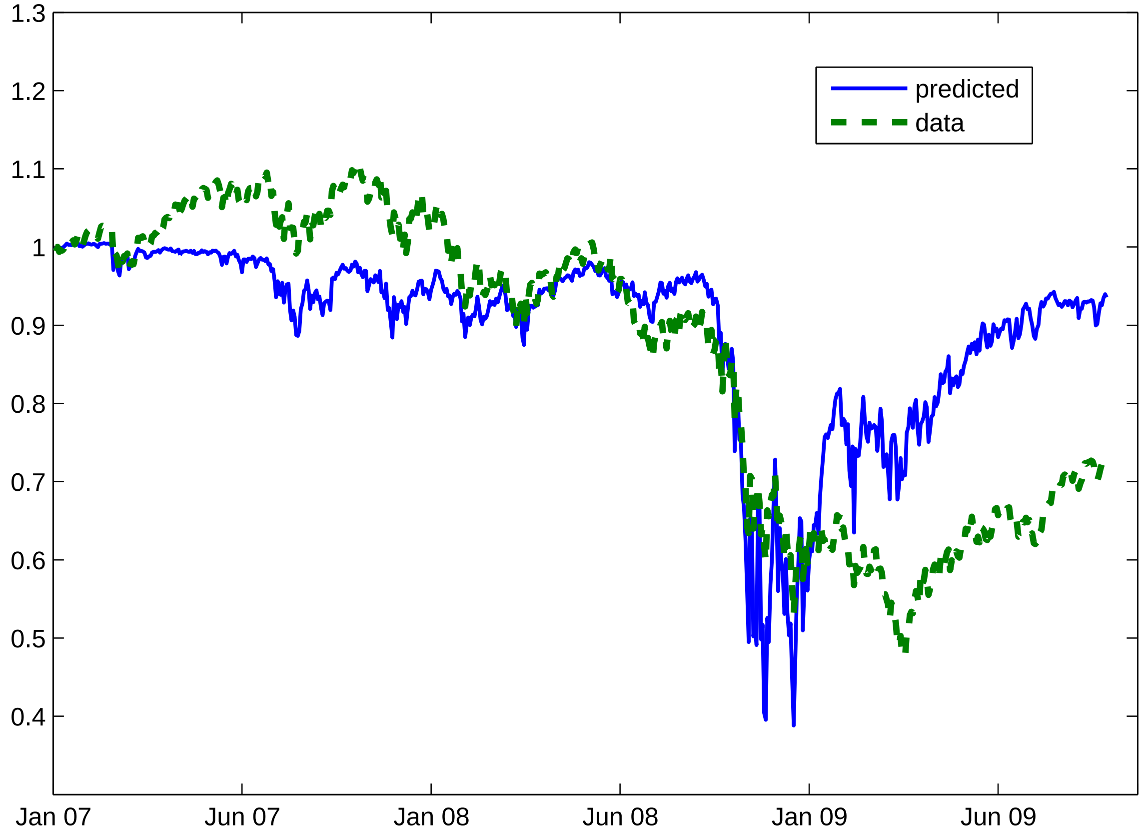

Figure 5: The figure shows the S&P 500 (dotted) vs the share price reaction predicted from the change in volatility.

5.2 Stock Returns

Next, we turn to the second issue of essence in this paper and we ask: can the sharp increase in volatility during the crisis "explain" the large negative stock returns over this period? In other words, how much of the negative stock returns observed over the sample period were due to an increase in risk premium? To shed light on this, I compute the return to the stock market conditional upon the realized volatility path as implied by the VIX index.

Figure 5 plots the expected stock price conditional upon the VIX, along with the actual index[11] Although the correlation is not perfect, the two time series share some strikingly common patterns:

- The predicted and actual stock price reactions are roughly similar. The model implies close to a 61% drop from the initial value compared to a 52% market decline.
- The model implies a market bottom on 20/11/2008 while the actual bottom occurred on 03/03/2009

The difference between what the market did and the model prediction is evident over the Dec 2008 - March 2009 period. Since volatility went down dramatically, prices should have increased but instead they decreased. The fact that the actual stock prices moved opposite of what is predicted by the decrease in volatility is interpretable as a negative shock to aggregate consumption in the model. Thus, to see if the model is consistent with the data we need to consider the actual macro economy over the Jan 2008 - March 2009. The US economy contracted 5.4 and 6.4% in the last and first quarters of 2008 and 2009 in what is widely regarded to be the worst recession since the Great Depression. Our model predicts that any change in the stock prices that is not volatility driven should be due to real dividend shocks. In looking at figure 5 we see that the eventual gap between the actual stock price and the volatility-predicted stock price is about 20%. Thus, the model is

[11] Using 7, the conditional expected stock return is computed as $r_{0,t} = B_{v,d}(V_t - V_0) = B_{v,d}/\beta_v(VIX_t^2 - VIX_0^2)$. Both series are normalized to $P_0 = 1$.

consistent with the data if the aggregate corporate losses were about 20% over this period, which seems plausible.

6 Concluding Remarks

This paper studies the large difference between the actual and options-implied volatility of stock returns. The difference between the two, the so-called volatility premium, is known to generate large returns to issuers of volatility-sensitive assets such as options. The theoretical equilibrium model considered in this paper does indeed produce a difference in the two volatility measures on the same order of magnitude as observed in the data. The key to constructing an equilibrium model in which volatility shocks have a sufficiently high market price of risk to generate the premium is the use of long-run risk equilibrium, coupled with a highly persistent volatility process.

The model delivers very high options returns, especially when volatility is high. This does not imply that investors who sell volatility earn risk-reward ratios that are substantially above those of other asset classes. The problem facing investors is that trading volatility is very risky. Thus, even if the mean returns on certain options classes are very high, so are the risks. The empirical options literature reports Sharpe ratios ranging from about one half to one. The model, at estimated parameters, implies an unconditional Sharpe ratio of about 0.48 for selling volatility. Thus, the model delivers a Sharpe ratio in the low range of what has been found in the empirical studies of options returns.

There are two possible reasons why some studies may have found higher Sharpe ratios. First, there is significant sampling variability in these reward-to-risk ratios because they are estimated over relatively short sample periods. Thus, the occurrence, or lack thereof, of significant market turmoil will influence the estimates. Second, it is possible that the model underestimates the premium, or overestimates the risks involved.

There are several stylized facts about volatility which is not incorporated into the model. First of all, several studies of the dynamics of stock market volatility suggest that volatility has a long-run-component. For example, Chacko and Viciera (2005) find that volatility is significantly more persistent when estimated using coarsely sampled data. Models such as those of Bates (2000) and Chernov, Gallant, Ghysels, and Tauchen (2003) provide an approximation to long memory models by specifying volatility as a two-factor process.[12] Clearly, by building models with higher long-run persistence in volatility, it is possible to attribute even higher risk premia to volatility shocks.

There are several ways in which one can argue that the current model does not yield a realistic representation of either the macro-economic environment or the asset price implications. For example, in this model the term structure of interest rates depends only on the macro-volatility factor. It is easy to add additional factors such as expected consumption growth, or expected inflation growth. As for the macro-economic realism, a few notes are in order. First, there is significant evidence of time-variation in the volatility of real consump-

[12] The term long-memory typically refers to processes where the autocorrelation function decays at a slower rate than exponential, as in one-factor models. Two-factor models asymptotically decay at an exponential rate. Models of fractionally integrated variance have slower decay rates.

tion growth. For example, using annual consumption growth data collected over a period of a hundred and fifteen years, GARCH model estimates indicate substantial persistence and time variation in the volatility[13]. It is possible to obtain volatility risk premia that are higher in models that have either multiple volatility factors, or have additional risk factors which depend linearly or non-linearly on volatility as in, for example, the Bansal & Yaron (2004) model. The possibility of constructing a unified consumption based pricing model that successfully explains the conditional movements in macro time-series and different financial assets (stocks, bonds, and derivatives) remains an extremely challenging but interesting agenda for future research.

[13]While not reported in detail here, I fitted GARCH (1,1) to annual consumption data and compared the fitted values to data on annual real consumption growth simulated from the model. The first order volatility persistence in the actual data was found to be about 0.96 which compares to 0.3 in the simulated data. Sampling variability is substantial, making inference difficult.

Appendix

A VIX as an Approximation to Risk Neutral Volatility

The VIX index is widely believed to be a *model free* estimate of the option-implied standard deviation of logarithmic returns. The squared VIX is computed as a discrete approximation to

$$VIX^2 = \frac{2e^{rT}}{T}\int_0^\infty \frac{Q(T,K)}{K^2}dK \tag{23}$$

where $Q(T,K) = \min(P(K), C(K))$ is the minimum of the Put and Call prices with strike K. In recent work, Martin (2011) points out that the VIX index is itself does not equal the variance of the log-return under the risk neutral measure when the return distribution departs from log-normality. In particular, Martin shows that the squared VIX depends on higher order cumulants in addition to the variance.

So how large is the error documented by Martin? Figure 6 plots the theoretical variance of the one month log-stock return (annualized), along with the theoretical VIX index as computed through equation (23). The plot shows *model-implied* quantities. The theoretical VIX index is computed from theoretical options prices from my model using parameter values in Table 2 over a fine grid of strike prices. Since the VIX index is itself a function of macro-variance V_t, the figure computes the errors across a grid of initial values of V_t. The

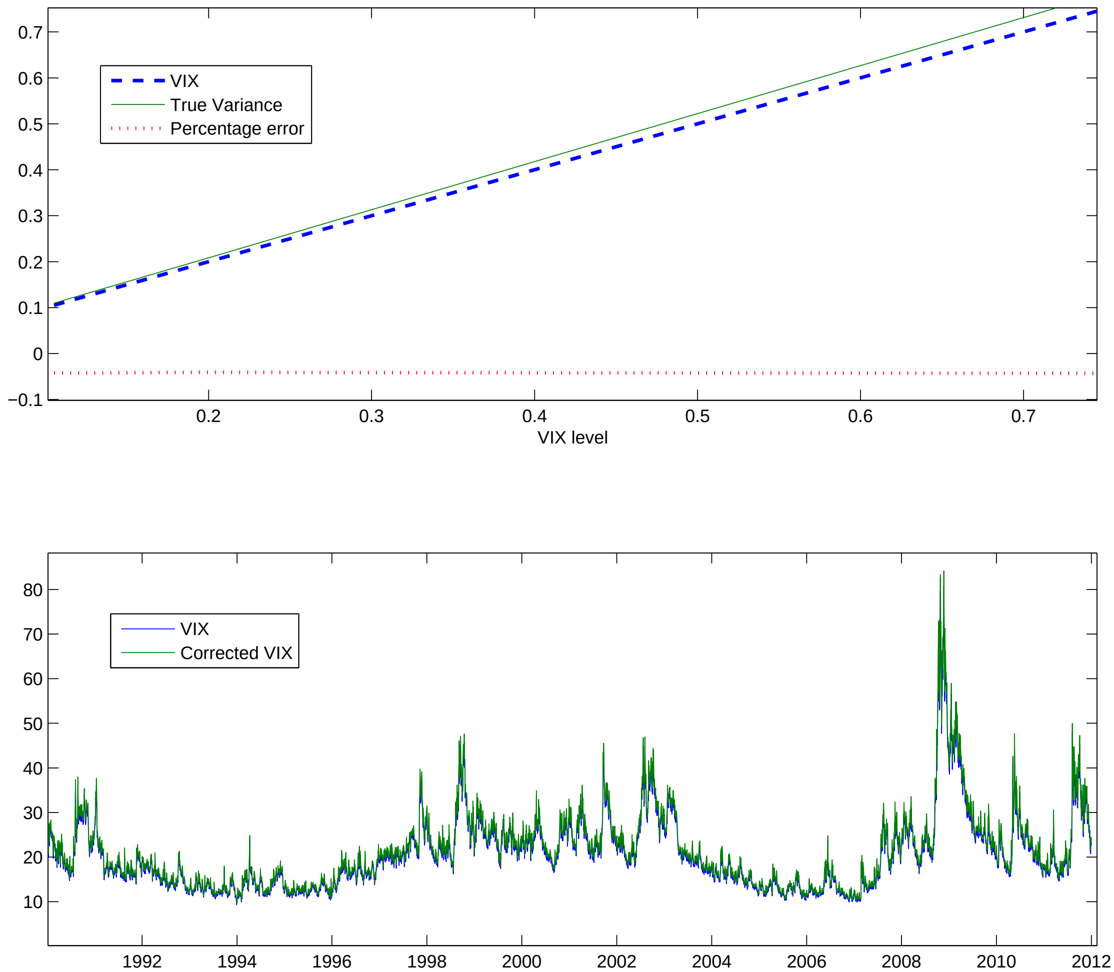

Figure 6: VIX vs. actual risk neutral variance using the theoretical option pricing model.

figure shows that the errors are approximately constant as a percentage of the VIX level. The errors range between -4.1% and -4.2% of the VIX value. For example, when the VIX is at 20, it is under-estimating the true risk neutral variance which is about 20.82. The errors are (approximately) a constant multiple of the level of the VIX index.

Thus, two insights emerge from this exercise. The first is that the error can easily be accounted for using a model. The second is that is that the error in the VIX is indeed small. To put it in perspective, consider the fact that the annualized standard deviation of the daily percentage changes in the VIX index itself is about 100%. Thus, the error is swamped by volatility. Visually, this can be represented by plotting the actual VIX index alongside with the corrected one, as in the bottom half of figure 6. As can be seen the two are virtually indistinguishable.

These conclusions are based on a specific model. To understand how our particular model impacts the computation of the error, consider instead the jump diffusion model of Merton (1976). Depending on parameter values, Merton's model will give errors ranging from about negative 1.7% to 3.4%. Thus, Merton's model yields a smaller but still negative bias. The smaller magnitude of bias in Merton's model is likely due to the behavior of the left tail of the risk neutral density which is decaying at a more rapid rate in Merton's model than the model considered here.

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