Dynamic Present Values and the Intertemporal CAPM

Bjørn Eraker
Wenyu Wang

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Paper Summary

Two parts

- ICAPM critique
  - i We show that the ICAPM is inconsistent with dynamic present value computations unless one makes unreasonable implicit assumptions about future cash flows
  - ii Implicit assumptions are present in (almost) all empirical studies of the conditional ICAPM

  This also applies to the discrete time *conditional CAPM*.

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  - ii Implicit assumptions are present in (almost) all empirical studies of the conditional ICAPM

  This also applies to the discrete time *conditional CAPM*.
- Derive new dynamic CAPM which relaxes the implicit assumptions
  - i Endogenizes risk feedback
  - ii Gives closed form dynamics for affine/power utility combo
  - iii Model produces empirically realistic volatility feedback/ risk premium

Outline of Talk

- ICAPM critique
- Derive the alternative model from PV
- Some empirical results (not in paper)

The ICAPM critique: Why?

Large literature on empirical tests ICAPM/conditional CAPM. This literature seeks to test the model

$$r_{i,t+1} = r_{f,t} + \beta_{i,t}(E_t(r_{m,t+1}) - r_{f,t}) + \epsilon_{i,t+1} \tag{1}$$

for some arbitrary asset *i* and typically

$$r_{m,t+1} = r_{f,t} + \gamma\sigma^2_{m,t} + \epsilon_{m,t+1} \tag{2}$$

for the market portfolio where $\sigma_{m,t}$ is a conditional market volatility measure, ϵ are shocks independent of the innovations in the expected returns.

Some conditional CAPM empirical papers:

Merton (1980), French, Schwert & Stambaugh (1987), Bollerslev, Engle & Wooldridge (1988), Gibbons & Ferson (1985), Engle, Lilien & Robins (1987), French, Schwert & Stambaugh (1987), Nelson (1988), Harvey (1989), Engle, Ng & Rothschild (1990), Harvey (1991), Ng (1991), Bodurtha & Nelson (1991), Ferson & Harvey (1993), Glosten, Jagannathan & Runkle (1993), Whitelaw (1994), Evans (1994), Moskowitz (2003), Guo & Whitelaw (2006), and Bali & Engle (2010, 2010), among others.

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The ICAPM critique: Background (cont')

Merton (1973) **assumes**

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_t dB_t \tag{3}$$

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So prices do not respond to changes in expected returns in Merton's model.

Hellwig's Critique

In an upublished working paper, Hellwig (1977) "On the Validity of the Intertemporal Capital Asset Pricing Model" critiques Merton's ICAPM. Hellwig points out that the exogenous price assumption is incompatible with certain cash flow and utility assumptions.

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Trivial example: Risk neutrality, normally distributed future cash flows (dividends), constant investment opportunity set. This gives normally distributed prices.This is incompatible with log-normality, as assumed in (3).

Hellwig writes

> *Merton's analysis of capital market equilibrium is unsatisfactory: he determines equilibrium prices by assumption rather than by demand and supply. First, he derives the portfolio behavior under the assumption that asset prices follow a log-normal process. Then he considers the implication of market clearing for average rates of return under a log normal process. He fails to verify that the assumption of log-normality itself is compatible with market clearing.*

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We are concerned with the dynamics of Merton's model.

Three period example

Consider a three period economy. Consume a risky amount $\tilde{x}$ at date 2. Trade at dates 0 and 1.

The investor receives news about the terminal payoff, $\tilde{x}$ and the next period's expected return μ_2 at time 1. The relative change in the expected return, μ_2/μ_1 and the expected terminal payoff $E_1(\tilde{x})/E_0(\tilde{x})$ are random variables as of time 0.

Theorem

Assume that the price process satisfies $Corr(R_1, \mu_2) = 0$*, its expected return process* μ_t *is random, in steady state* $E_0\mu_2 = \mu_1$*, and that the price process is consistent with present value computations. Then*

$$Corr\left(\frac{E_1(\tilde{x})}{E_0(\tilde{x})}, \frac{\mu_2}{\mu_1}\right) > 0. \tag{4}$$

The theorem states that if one rules out *a priori* that shocks to expected rates of return have impact on prices $\mathrm{Corr}(R_1, \mu_2) = 0$, shocks to expected rates of return, μ_t, **must** be positively correlated with shocks to the terminal expected value, $E_t(\tilde{x})$, for the price process to be consistent with equilibrium.

Corollary

Under the same assumptions as in Theorem 1, if

$$Corr\left(\frac{E_1(\tilde{x})}{E_0(\tilde{x})}, \frac{\mu_2}{\mu_1}\right) = 0, \tag{5}$$

the price process is inconsistent with present value computation.

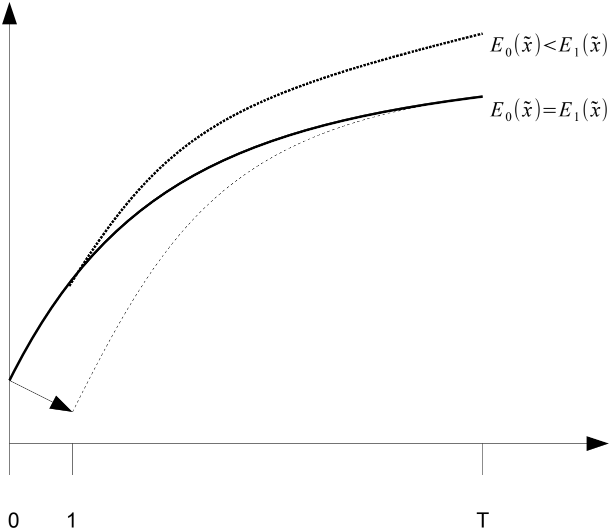

Figure 1: Possible price paths with and without volatility feedback.

Importance of endogenous volatility feedback

The condition $\text{Corr}(R_1, \mu_2) = 0$ rules out volatility feedback. This is equivalent to assuming

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- Empirically strong negative correlation between volatility changes and prices. Will allow sharp inference about γ.
- Empirical specifications of the relationship between volatility and returns will be misspecified unless we account for the contemporaneous effect

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- Utility of terminal wealth $u(\tilde{x}_T)$ (special case of Merton '73)

General Result

Proposition 2. *The equilibrium market price price P_t of the aggregate wealth claim $\tilde{x}_T$ is given by*

$$P_t = \frac{E_t\left\{u'(\tilde{x}_T)\tilde{x}_T\right\}}{E_t\left\{u'(\tilde{x}_T)\right\} R_{t:T,f}}, \qquad \forall t \in (0, T). \tag{6}$$

The price of an arbitrary asset, $P_{t,i}$, with terminal payoff $\tilde{x}_{T,i}$ in infinitesimal supply is

$$P_{i,t} = \frac{E_t\left\{u'(\tilde{x}_T)\tilde{x}_{T,i}\right\}}{E_t\left\{u'(\tilde{x}_T)\right\} R_{t:T,f}}, \qquad \forall t \in (0, T). \tag{7}$$

These price processes obtain irrespectively of the frequency of trade and portfolio rebalancing.

Remarks:

- Follows from market clearing and FOC

$$E_t\left\{u'(\tilde{x}_T)R_{m,t:T}\right\} = E_t\left\{u'(\tilde{x}_T)R_{r,t:T}\right\} \tag{8}$$

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- Eqns. (6) and (7) can be used directly to compute equilibrium asset prices
- Same as in dynamic programing solution. Time consistent.

Example Model

Assume power utility of terminal wealth $u'(\tilde{x}_T) = \tilde{x}_t^{-\gamma}$ and exogenous terminal wealth $\tilde{x}_T$ given by

$$\frac{d\tilde{x}_t}{\tilde{x}_t} = \mu dt + \sigma_t dB_t^w, \tag{9}$$

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \sigma_v \sigma_t dB_t^v + \xi_t dN_t \tag{10}$$

$$\xi_t \sim \text{Exp}(\mu_\xi) \tag{11}$$

where we assume the standard affine state dependent jump arrival intensity, $l(\sigma_t^2) = l_0 + l_1 \sigma_t^2$ where l_0, l_1 are parameters.

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Notice that the infinite horizon limit is just

$$\lambda_\sigma = -\frac{\gamma}{\kappa}$$

reminiscent of **long-run-risk.**

Some intuition:

- The volatility feedback can be large, and generates substantial negative correlation between changes in vol and returns even for small γ.
- Model does not yield ICAPM unless vol-of-vol is zero.
- λ_0 is the unconditional expected return
- When volatility is high ($\sigma_t^2 > E(\sigma_t^2)$), then $E_t(d\sigma_t) < 0$ because of mean reversion. The conditional risk premium is

$$E_t(\ln R_t - r_{f,t})dt = \lambda_0 dt - \frac{\gamma}{\kappa} E_t(d\sigma_t^2)dt > \lambda_0 dt$$

so the conditional risk premium is greater than its long run mean.

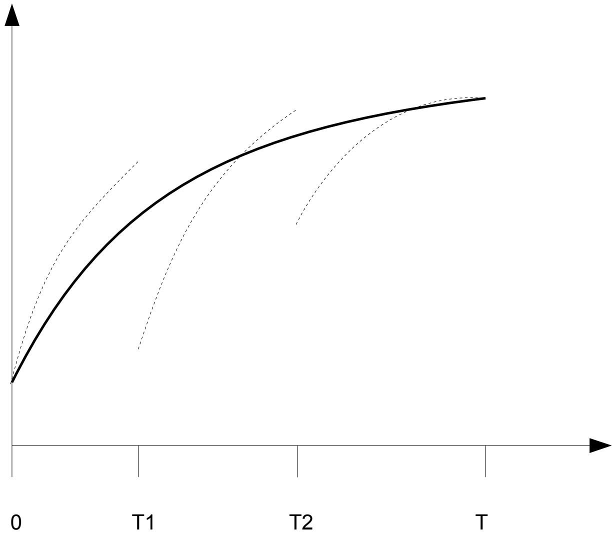

Figure 2: Expected price path. Unconditionally expected path (solid) vs. path with volatility jumps at dates $T1$ and $T2$ (dashed) .

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- Thus: Our model can be seen as a special case of Epstein-Zin when ψ is large (investors do not care about the timing of their consumption flows)
- However: No Campbell-Shiller approximations.

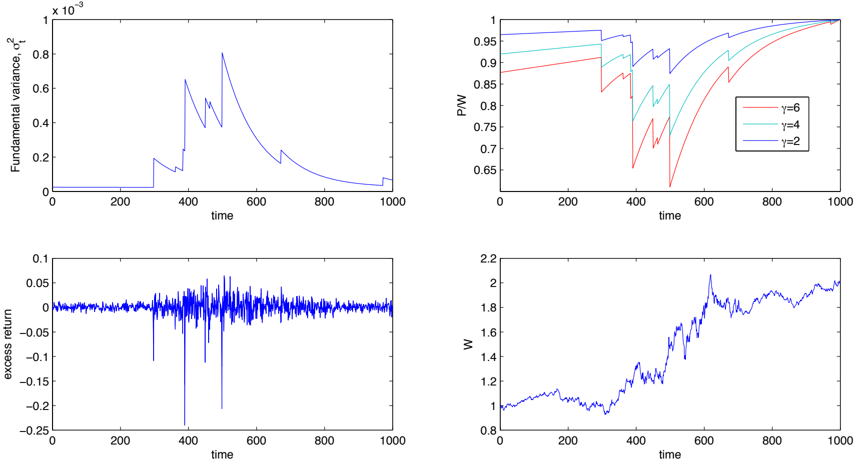

Figure 3. Simulated sample paths for the example model.

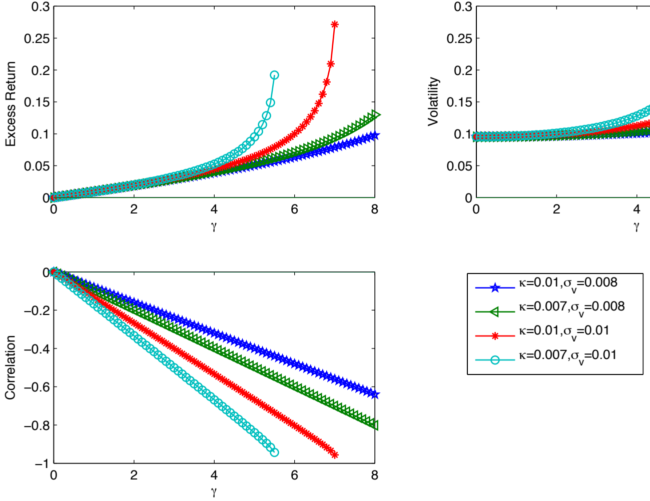

Figure 3. Excess returns, volatility, and correlation. Diffusive case.

Revisiting the ICAPM

ICAPM is

$$d \ln P_t = r_{f,t} dt + \gamma \sigma_t^2 + \sigma_t dB_t$$

Our model is

$$d \ln P_t = r_{f,t} dt + \lambda_0 dt + \lambda_\sigma d\sigma_t^2 + \sigma_t dB_t$$

- Merton's model is missing the volatility feedback term $\lambda_\sigma d\sigma_t^2$.
- ICAPM does not hold in our model. With stochastic vol, there are always at least two priced risk factors. Expected returns are not given by ICAPM.

Estimating γ

- Standard ICAPM: Identification of γ from the predictive relationship
$$r_{m,t+1} - r_{f,t} = \gamma \sigma_t^2 + \epsilon_{t+1}$$
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Weak correlation shows in many insignificant estimates in previous ICAPM studies.
- In our model we can identify γ from the contemporaneous relationship $\Delta \ln S = \text{const.} - \gamma/\kappa\Delta\sigma_t^2 + \text{error}$. This gives sharp identification.

Table 1: Parameter Estimates

The table reports estimates of the posterior distribution of parameters in our model. Estimates are based on MCMC estimation of the parameters along with the unobserved volatility V using daily data on the Value Weighted CRSP and S&P 500 excess returns.

	S&P 500, 1926-2012 , T=22,780			
	$\theta \times 1000$	κ	$\sigma_V \times 1000$	γ
Mean	0.077	0.009	0.961	5.777
Std	(0.007)	(0.001)	(0.030)	(0.663)
	CRSP-VW, 1963-2012, T=12,273			
	$\theta \times 1000$	κ	$\sigma_V \times 1000$	γ
Mean	0.060	0.011	0.884	7.180
Std	(0.008)	(0.002)	(0.061)	(1.282)

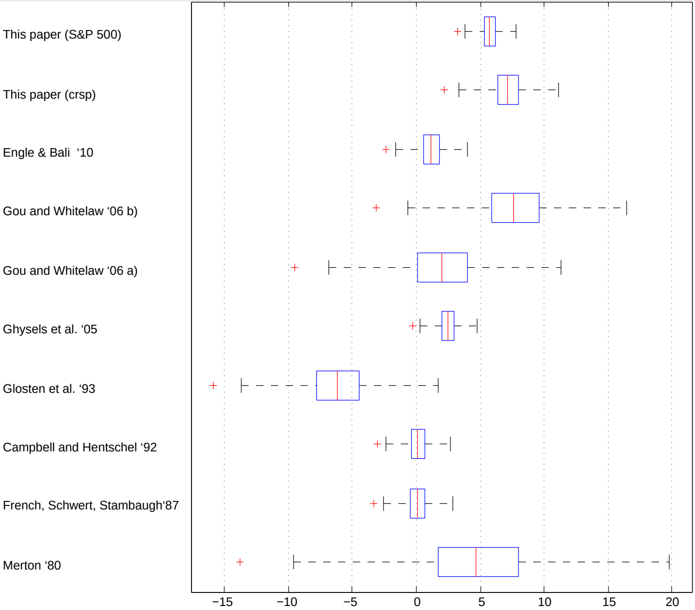

Figure 4. Sampling/ posterior distributions for γ.

Table 2. Performance of ICAPM Estimators of γ

We report estimates of γ using three estimators: OLS on RV as in French, Schwert and Stambaugh, an estimator based on full knowledge of the volatility process (OLS-V), and Garch(1,1) in mean. The DGP is our structural model.

	OLS-RV	OLS	M-GARCH
Panel A: $\theta = 0.0065^2, \kappa = 0.0075, \sigma_V = 7.8405e-04, \gamma = 5$			
mean	3.69	4.17	0.73
rmse	(1.69)	(1.82)	(0.37)
bias	-26.20	-16.69	-85.33
Panel B: $\theta = 0.0065^2, \kappa = 0.0126, \sigma_V = 9.92e-04, \gamma = 6.86$			
mean	4.35	5.29	0.75
rmse	(1.46)	(1.64)	(0.38)
bias	-36.62	-22.89	-89.06

Where does the bias come from?

OLS estimators work from the assumption that γ can be estimated as the slope coefficient β in the predictive regression

$$r_{t+1} - rf = \alpha + \beta \mathrm{Var}_t(r_{t+1}) + \epsilon_{t+1} \tag{14}$$

- Well known: Errors-in-variables bias in when conditional variance measures have measurement errors
- Here: Also bias because ICAPM does not hold. We can compute this bias explicitly

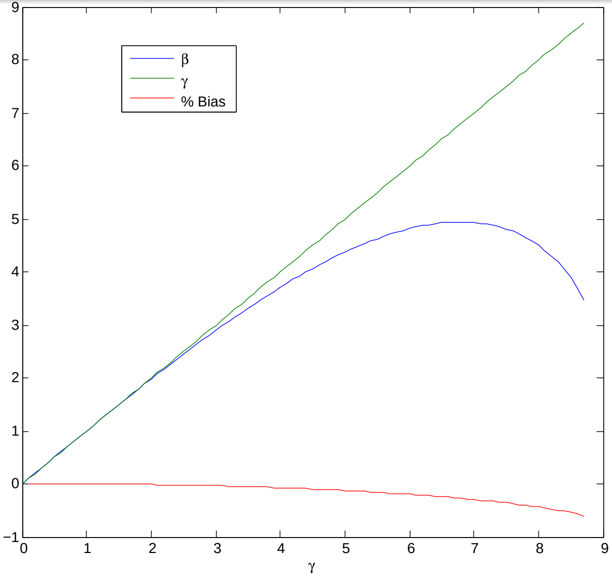

Figure: Assuming our model constitute the true DGP, we compute the bias in the estimate of γ from regressing returns on conditional variance.

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- Generates at least two priced factors (cash flow and volatility risk)