Vasicek and CIR Models

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The Vasicek Model

The Vasicek model (Vasicek 1978) is one of the earliest no-arbitrage interest rate models.

It

- is based upon the idea of *mean reverting* interest rates
- gives an explicit formula for the (zero coupon) yield curve
- gives explicit formulaes for derivatives such as bond options
- can be used to create an interest rate tree

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AR(1) Processes

An *autoregressive process of order 1* (AR(1)) is a *stochastic process* of the form

$$Y_t = a + bY_{t-1} + \epsilon_t$$

where a and b are constants, ϵ_t is a random error term, typically

$$\epsilon_t \sim N(0, \sigma^2)$$

Mean Reversion

- If a process is mean reverting, it tends to revert to a constant, long run mean
- The speed of mean reversion measures the average time it takes for a process to revert to its long run mean
- Mean reversion is reasonable for interest rates - random walk makes no sense because it is economically unreasonable to think that interest rates can "wander of to infinity" or become arbitrarily large.
- The mean reversion in the AR(1) process is measured by b.
- So what is the typical speed of mean reversion in interest rates? Weekly data gives 0.9961 for 3m T-bills. This is *extremely high* persistence and indicates that interest rates are *close to* random walk...

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- b close to 1 means slow mean reversion
- $b = 1$ means that Y_t is a *Random Walk*. There is NO mean reversion!
- b close to zero means that there is rapid mean reversion.

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The Vasicek Model

The Vasicek model simply assumes that the short rate, r_t follows an AR(1),

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Alternative Representations

The model is sometimes written,

$$\triangle r_t = \kappa(\theta - r_{t-1})\triangle t + \sigma \triangle z_t \tag{2}$$

where

$$\triangle r_t = r_t - r_{t-1} \tag{3}$$

and $\triangle z_t$ is a normally distributed error term.
Using the definition of $\triangle r_t$, and set $\triangle t = 1$, we have

$$r_t = \kappa\theta + (1 - \kappa) r_{t-1} + \sigma \triangle z_t \tag{4}$$

which is an AR(1) if $a = \kappa\theta$ and $b = 1 - \kappa$.

We can also set $\triangle t$ to something different from 1. In this case we assume that

$$\triangle z \sim N(0, \triangle t)$$

Differential Notation

Tuckman writes

$$dr_t = \kappa(\theta - r_t)dt + \sigma dw_t$$

This is a *stochastic differential equation* where *dw* is *Brownian motion*. You should interpret it to mean

$$\triangle r_t = \kappa(\theta - r_{t-1})\triangle t + \sigma \triangle z_t \tag{5}$$

In other words, we interpret dr_t to mean $\triangle r_t = r_t - r_{t-1}$. Therefore, any time you see a the differential notation in Tuckman, translate by equating to the *first difference*.

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Vasicek Zero Coupon Curve

Let $B(t, T)$ denote the time t price of a zero coupon bond with maturity date T. It is

$$B(t,T) = \exp(-A(t,T)r_t + D(t,T)) \tag{6}$$

$$A(t,T) = \frac{1-e^{-\kappa(T-t)}}{\kappa} \tag{7}$$

$$D(t,T) = \left(\theta - \frac{\sigma^2}{2\kappa}\right)[A(t,T)-(T-t)] - \frac{\sigma^2 A(t,T)^2}{4\kappa} \tag{8}$$

In other words, the Vasicek model gives us an exact expression for the value of a zero coupon. We can compare this to other models for zero coupon bond prices, such as the *cubic spline* model and *Nelson-Siegel*.

Properties of the Vasicek term structure

Lets take some example parameter values to see how the Vasicek term structure behaves. Reasonable values are

$$\kappa = (1 - 0.9961) \times 52 \approx 0.2,$$

$$\sigma = 0.0025 \times \sqrt{52} \approx 0.018,$$

(180 basis p/ year) and

$$\theta = 0.05$$

(the average risk free rate).

Effects of mean reversion

- The smaller the mean reversion in interest rates, the more we expect interest rates to remain close to their current levels.
- If we expect interest rates to change very slowly, long dated bonds tend to move in parallel to short.
- Thus, higher (lower) speed of mean reversion ($\kappa = 1 - b$) lead to a steeper (less steep) term structure because *the term structure depends on future expected short rates* (illustrated on next slide)

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Vasicek Bond Options

Consider an option on a zero coupon. The underlying bond has maturity T and the option matures at T_0 (we assume $T_0 < T$ why?).
The value of a call option with strike K is

$$C = B(t, T)N(d_1) - KB(t, T_0)N(d_2) \tag{9}$$

where d_1 and d_2 have expressions that mimic those in the Black-Scholes model.

One of the features of the Vasicek model is that interest rate changes have constant volatility, σ, no matter what happens in the economy.

It can be useful to relax this assumption for two reasons

- We have empirical evidence suggesting that interest rate changes are *more* volatile when the level of interest rates are high
- As we will show, interest rates will remain positive..

The model is

$$dr_t = \kappa(\theta - r_t)dt + \sigma^* \sqrt{r_t} dW_t$$

Thus, the CIR model has the same *drift* term as the Vasicek model, but the volatility is

$$\text{Std}(dr_t) = \sigma^* \sqrt{r_t}$$

This is an example of a *conditional* volatility: when the short rate r_t is high the volatility of interest rate changes is high and vise versa.
The parameter σ^* no longer represents the volatility of interest rate changes. It is just a parameter...

Choosing σ^*

Last lecture we argued that the volatility of short rates changes are some 180 basis points per year. In the CIR model, volatility is time varying.

It is reasonable therefore to choose σ^* such that the average volatility under the CIR model is approximately equal to 180 basis point (0.018).

We can do this by solving

$$0.018^2 = E(\mathrm{Var}(dr_t)) = \sigma^{*2} E(r_t) = (\sigma^*)^2 0.05$$

or

$$\sigma^* = \sqrt{\frac{0.018^2}{0.05}} \approx 0.08$$

Random Simulation

Suppose we draw the random shocks, e_t out of a hat for $t = 1, .., T$. All the numbers in the hat are normally distributed with mean zero and variance one.

We can now use these numbers to simulate what would happen to future interest rates. We use the equation

$$r_t = r_{t-1} + \kappa(\theta - r_{t-1})dt + \sigma\sqrt{dt}e_t \tag{10}$$

for some dt.

In the last lecture I argued that $\kappa = 0.2$, $\theta = 0.05$ and $\sigma = 0.018$ represent reasonable values describing the year-to-year behavior of US interest rates.

- Get normally distributed random numbers using *NORMSINV(RAND())*
- Use equation (**??**) to compute all future interest rates as functions of κ, θ, σ and the time step *dt*.

See that

- When interest rates are high (low), the subsequent rates tend to decrease (increase) toward the long run average rate, θ.
- When κ is high, rates mean revert more quickly
- Interest rates can become negative in the Vasicek model

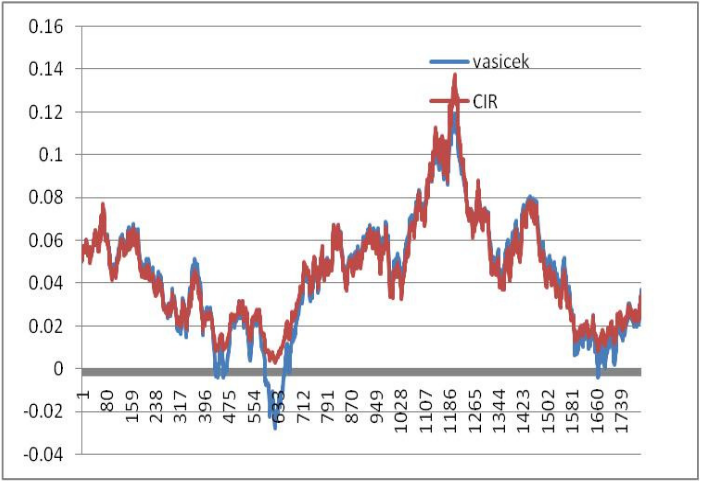

Figure : Simulated Vasicek and CIR interest rates. Shocks are the same.

Notice that the simulated CIR path is *positive* while becoming negative for the Vasicek model.
This is not a coincidence: In theory, all CIR paths should remain positive....
Our excel sheet for simulation may produce negative values still because our simulations are approximations to the continuous time model in CIR....

CIR term structure

Prices of zero coupons with maturity T are given by

$$P(T) = A(T)e^{-B(T)r_0} \tag{11}$$

where

$$\begin{aligned} A(T) &= \left[\frac{2he^{(\kappa+h)T/2}}{2h+(\kappa+h)(e^{hT}-1)}\right]^{\frac{2\kappa\theta}{\sigma^{*2}}} && (12) \\ B(T) &= \frac{2(e^{hT}-1)}{2h+(\kappa+h)(e^{hT}-1)} && (13) \\ h &= \sqrt{\kappa^2+2\sigma^{*2}} && (14) \end{aligned}$$