Interest Rate Trees

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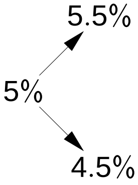
- Suppose we have a simple economy where *spot* rates can go either up or down by a fixed amount over the next six months.
- Remember: the spot rate is the semi-annually computed ytm on a zero.
- For example :



Lets consider a bond with \$ 1000 face with a six month maturity. At time 0 it is worth

$$\frac{1000}{1+\frac{0.05}{2}} = \$975.61$$

If, on the other hand, we are in the "up-state six months from now, a six month bond is worth

$$\frac{1000}{1+\frac{0.055}{2}} = \$973.24$$

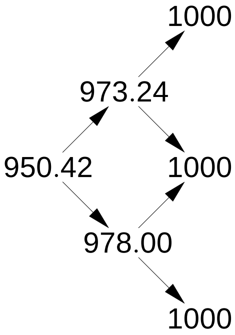
Of course, in the down state we have

$$\frac{1000}{1+\frac{0.045}{2}} = \$978$$

Moreover, if we assume that the *one year* spot rate is 5.15%, we get the time zero price of the one year zero as

$$\frac{1000}{(1 + \frac{0.0515}{2})^2} = \$950.42$$

We have the following tree for the price of the one year



Risk-Neutral Probabilities

- Suppose we were to assume that the probability of an up/down move were 50/50.
- In this case, we should simply compute the value of the one year bond at time zero as the expected future value of the bond, discounted at the risk free rate.
- If so, we get

$$\frac{\frac{1}{2}973.24 + \frac{1}{2}978}{1 + 0.05/2} = 951.82$$

which is different from the actual price of 950.42.

What is going on? Clearly, the market prices are inconsistent with the probability of $1/2$ for an up/down move.
Let's see what the probability of an up/down move must be....
Let p denote the probability of an up-move, $1 - p$ a down move
It must be that

$$\frac{p \times 973.24 + (1 - p) \times 978}{1 + 0.05/2} = 950.42$$

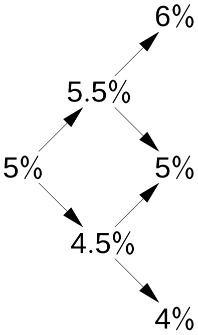
Solving for p gives $p = 0.8024$.

About Risk Neutral Probability Measures

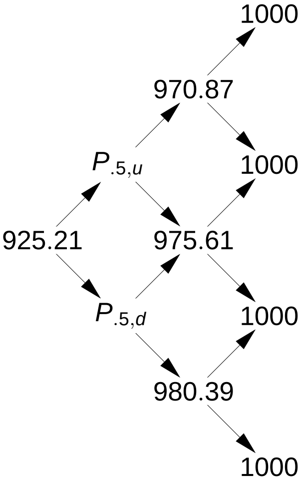
- Risk neutral probabilities may not equal the actual probability of an up down move (called the "objective" measure)
- Risk neutral probabilities are implied *by absence of arbitrage*
- Notice that we obtained the risk neutral probabilities simply from knowing the spot rates and the possible future interest rate path
- We can interpret risk neutral probabilities as *risk adjusted* probabilities
- Complete markets = all claims can be hedged.
- The fundamental theorem of asset pricing: If markets are *complete* and there are no arbitrage opportunities, there exists a unique risk neutral probability measure. If there exists a unique risk neutral probability measure, markets are complete and there are no arbitrage opportunities.

Multiple Periods

Suppose we extend the tree to 1.5 years:



Assume further that the 1.5 year spot rate is 5.25%. In this case we recover the price of a 1.5 year zero as



Here of course, we use the last nodes in the interest rate tree to compute the prices at time 1.5. For example

$$970.87 = \frac{1000}{1 + 0.06/2}$$

and so on.
Similarly, on date 0 the value of the bond is just

$$\frac{1000}{(1 + 0.0525/2)^3} = 925.21$$

We can similarly show that the probabilities of rates going up between 6M and 1YR is

$$q = 0.6489$$

(To derive this, solve two linear equations with unknowns $P_{.5,u}$ and q_u in the upstate, and $P_{.5,d}$ and q_d - the details are not important.)

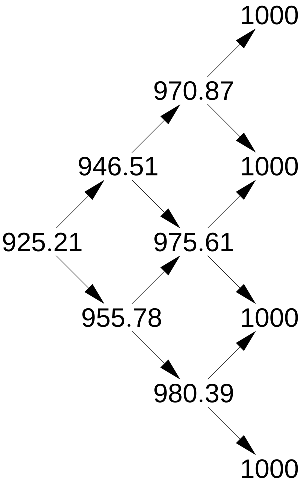
With $q = 0.6489$ we find that

$$P_{.5,u} = \frac{0.6489 \times 970.87 + (1 - 0.6489) \times 975.61}{1 + 0.055/2} = 946.51$$

and

$$P_{.5,u} = \frac{0.6489 \times 975.61 + (1 - 0.6489) \times 980.39}{1 + 0.045/2} = 955.78$$

Thus, the whole tree for the 1.5 year zero coupon bond is



Using the tree

Let's value a bond coupon bond. Consider a 4% semi-annual Treasury with 1.5 year to maturity. We will use "backward induction."

You will get paid $2 every six months and 102 at $t = 1.5$. We

start from the last period and work backwards "state by state." Thus, in the up-up state (rate is 6% in year one), the coupon

bond is worth

$$\frac{102}{1 + 0.06/2} + 2 = 101.03$$

where the $+2$ represents that periods coupon payment. This is the value including the coupon.

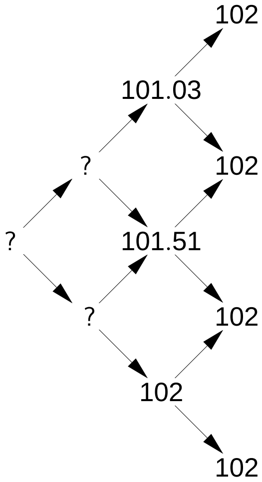
Similarly in the other states we find

$$\frac{102}{1+0.05/2}+2=101.51$$

and

$$\frac{102}{1+0.04/2}+2=102$$

We have filled in the values



We now find the 6 month prices.

In the up-state

$$P^c_{.5,u} = \frac{0.6489 \times 101.03 + (1 - 0.6489) \times 101.51}{1 + 0.055/2} + 2 = 100.49$$

and

$$P^c_{.5,d} = \frac{0.6489 \times 101.51 + (1 - 0.6489) \times 102}{1 + 0.045/2} + 2 = 101.45.$$

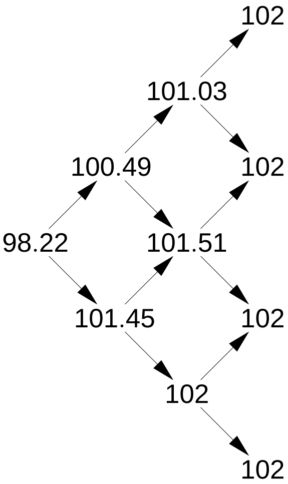
Note that we are using the same probabilities as we found before using the zero coupons.

Finally we find the price at time 0:

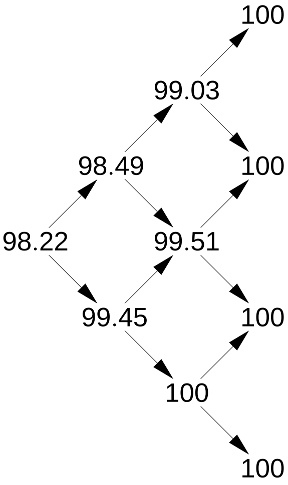
$$P_0 = \frac{0.8024 \times 100.49 + (1 - 0.8024) \times 101.45}{1 + 0.05/2} = 98.22$$

Note that we assume that the bond does not pay coupon at time 0.

The tree for the 2% coupon bond is now



Note that we can also write the tree without the coupon payments:



We prefer to write the tree without the coupons when valuing options on coupon bonds.

We can do the backward induction to get the bond price minus the coupon by computing

$$\frac{0.6489 \times 99.03 + (1 - 0.6489) \times 99.51 + 2}{1 + 0.055/2} = 98.49$$

and so on.

Application: Constant maturity treasury swaps

This contract pays the difference between the yield on some bond and a fixed rate times a notional amount. The book uses the example

$$\frac{1,000,000}{2}(y_{cmt} - 5\%)$$

where y_{cmt} is just the spot rate given in our interest rate tree. Thus, for the first period we get

$$\frac{1,000,000}{2}(5.5\% - 5\%) = 2,500$$

if rates go up, and

$$\frac{1,000,000}{2}(4.5\% - 5\%) = -2,500$$

if they go down.

Similarly, we find that three different states at time 1 produce payoffs,

$$\begin{aligned}
\frac{1,000,000}{2}(6\% - 5\%) &= 5,000 \\
\frac{1,000,000}{2}(5\% - 5\%) &= 0 \\
\frac{1,000,000}{2}(4\% - 5\%) &= -5,000
\end{aligned}$$

The value of the swap at time $t = 1/2$ in the up state is

$$\begin{aligned} & 2500 + \text{present value of expected future payoff} \\ = \; & 2,500 + \frac{0.6489 \times 5,000 + 0.3511 \times 0}{1 + .055/2} = 2,500 + 3,157.66 \\ = \; & 5,657.66 \end{aligned}$$

Similarly in the down state

$$\begin{aligned} & 2500 + \text{present value of expected future payoff} \\ = \; & -2,500 + \frac{0.6489 \times 0 + 0.3511 \times -5,000}{1 + .045/2} = -4,216.87 \end{aligned}$$

We now find the time 0 value of the swap as

$$\frac{0.8024 \times 5,657.66 + 0.1976 \times -4,216.87}{1 + 0.05/2} = 3,616.05$$

Note that all expectations are computed from the risk neutral probabilities - not the "objective" probabilities (1/2). If we had used the objective probabilities, the value of the swap would have been exactly..... ?

Pathwise Valuation

The standard method for valuation in trees is the backward induction method we have used. However, the CFA curriculum talks about pathwise valuation (section 43-3.6), so I will briefly describe this: In a "small" tree, take each possible path.

Compute the present value of a cash flow for each possible path.

Compute the probability of each path, and then value the security by averaging the NPVs for each path.

Consider the 18 month example from before. We only care about the rates in the first year, so we et that the possible paths are $\{U,U\},\{U,D\},\{D,U\}$ and $\{D,D\}$.
The NPVs for each path are

- UU : $1000 \times \frac{1}{1+0.05/2} \times \frac{1}{1+0.055/2} \times \frac{1}{1+0.06/2} = 921.8432$
- UD: $1000 \times \frac{1}{1+05/2^2} \times \frac{1}{1+0.055/2} = 926.3400$
- DU 930.8698
- DD 935.4329

The corresponding probabilities are

- UU : $0.8024 \times 0.6489 = 0.5207$
- UD: $0.8024 \times (1 - 0.6489) = 0.2817$
- DU $(1 - 0.8024) \times 0.6489 = 0.1282$
- DD $(1 - 0.8024) \times (1 - 0.6489) = 0.0694$

Using these probabilities multiplied by the NPVs on the previous page gives 925.2103 - the same value as with backward induction.

Further CFA notes

The CFA problems use annual trees.

There is a discussion of calibration to log-normal trees on p. 293. We talk about log-normal trees in connection with Black-Derman-Toy trees.

The CFA material is brief. You should be able to do practice problems connected to Reading 43, minus the problems on bonds with built-in options.

Concluding remarks

- Interest rate trees are simple devices for modeling the random movements in interest rates
- Interest rate trees are easy to adapt to pricing new derivative securities
- Famous models:
  - Original Salomon Bros model (no authors)
  - Ho-Lee
  - Black-Derman-Toy
  - Black-Karasinsky
  - Hull-White
- Shortcoming: Difficult to consider more than one source of randomness. For example, it is hard to make a model that allows for variation in both level and slope