Options

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Basics

Let $B(t)$ denote some bond price.
A bond call option with strike K pays

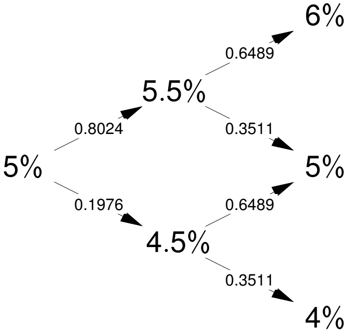
$$C(T) = \max(B(T) - K, 0)$$

at expiration time T of the option. Similarly, a put pays

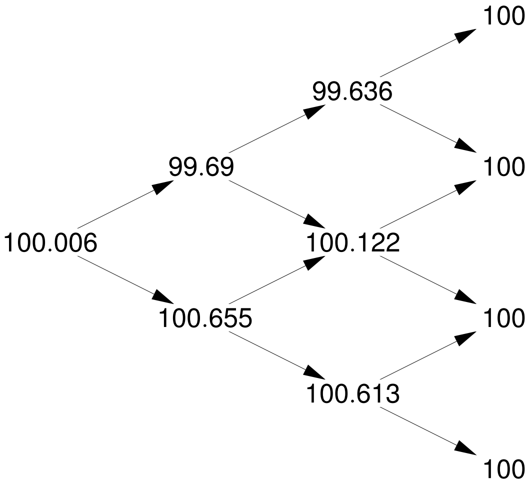
$$P(T) = \max(K - B(T), 0)$$

if executed at expiration.

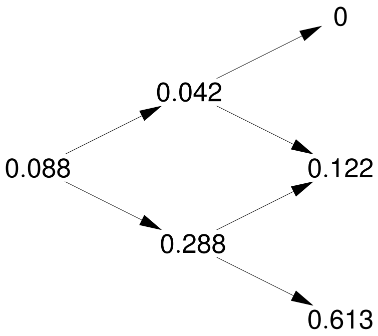
Let's start with a familiar interest rate tree



We can verify that the ex-coupon price of a 5.25%, 1.5 yr zero is



The price of a 1yr, $K = 100$, European Call is



Let's now consider an American option. We can now exercise at any point prior to expiration.

We need to see if it is better to exercise and get $B(t) - K$, or keep the option alive for all times t.

Yr 1 upstate: Bond (ex-coupon) is worth 99.636, which is less than the strike. Clearly not optimal to exercise.

Yr 2 downstate: Bond is worth 100.655 which gives 0.655 if exercised and 0.288 if kept alive. Optimal to exercise.

The time 0 value of the option if not exercised is

$$\frac{0.8024 \times 0.042 + 0.1976 \times 0.655}{1 + 0.05/2} = 0.159.$$

Now let's see if it is optimal to exercise.

If exercised, we get $100.006 - 100 = 0.006$ which is less than 0.1588 so we are better off holding on.

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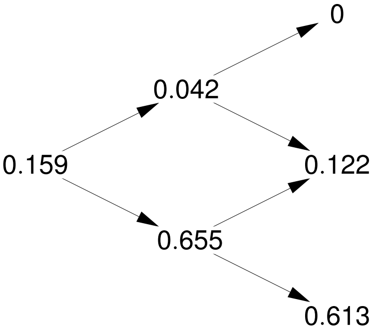
In the end we get the following path for the option:



Callable Bonds

Bonds that are callable have been used by the US govt historically, but there are no recent issues. The last ones were called recently.

The most common form of callable debt is *mortgages*, since all of us have the option to pre-pay our mortgage. Mortgages are complicated because more than just interest rate levels determine whether we exercise the option (pre-pay).

The agencies still issue callable bonds to hedge against mortgage pre-payment risk.

From the fannie web-site:

Callable debt is one of the most important financial tools Fannie Mae uses to match the duration of its liabilities to that of its mortgage assets when mortgages prepay. By issuing callable debt, the company gains protection against declining interest rates that tend to cause the mortgage assets of the portfolio to prepay more quickly. Fannie Mae can then redeem the company's currently callable debt to match the liquidations of the company's mortgage assets, thus keeping the duration of the company's assets and liabilities closely in line.

Pricing Callable Bonds

It's simple!

Let P_C, P_{NC} and C denote the prices of the callable, a standard non-callable with same maturity and coupon, and the option embedded in the callable (strike 100). Then

$$P_C = P_{NC} - C$$

Thus, it is easy to price a callable:

1. Price a hypothetical non-callable with same maturity, coupon
2. Price the imbedded option
3. The value of the callable is the value of the non-callable minus the option

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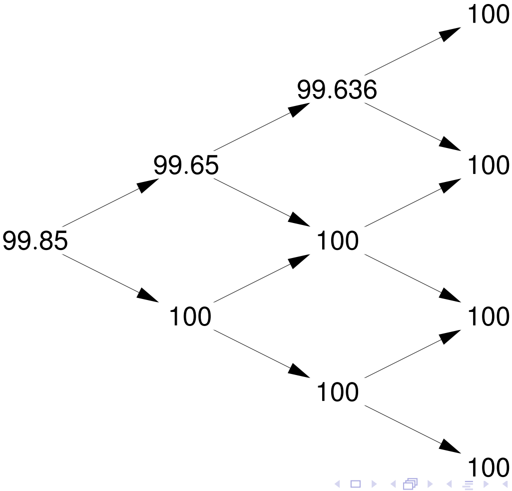
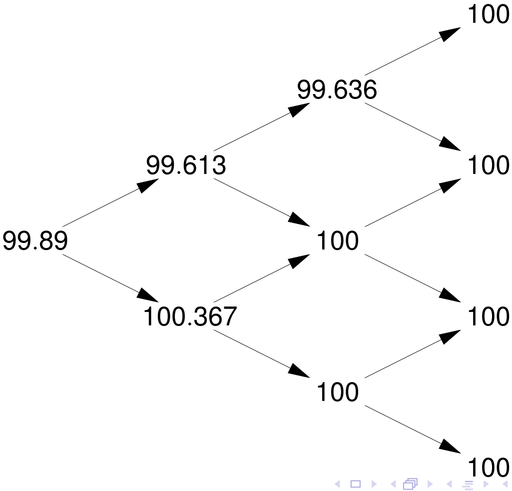
Let's look at the price path of a 5.25%, 1.5 year callable anytime:



Here is the path of a 5.25%, 1.5 year callable after one year (imbedded European option):



Comments:

- The callable bond always sell at par or less if callable anytime
- The callable bond may be worth more than par before the exercise date in the European case (bond must be called on specific date)

To see why the callable anytime must sell below or at par:

It is always the case that an option is worth at least as much as if immediately exercised

$$C(t) \geq P_{NC}(t) - 100$$

or

$$100 \geq P_{NC}(t) - C(t) = P_C(t)$$

Yield-to-Call

Is the bond's yield if held until the first call date and assuming that the bond is called on that date.

In the case that the 5.25 is callable in year 1 only, we solve

$$99.89 = \frac{5.25/2}{1 + y/2} + \frac{100 + 5.25/2}{(1 + y/2)^2}$$

to find the yield-to-call, y. We find

$$y = 5.36\%$$

Swaptions

An owner of a swaption has the right (option) to enter a swap contract at some future date(s).

To value a swaption,

- Break the swap into fixed and floating rate notes.
- Remember that the value of the floating leg is always par.
- Thus, an option on the swap is essentially just an option on the fixed rate note imbedded in the swap contract.

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Caps and Floors

A *caplet* is an option written directly on a reference rate. For example if LIBOR, denoted $L(t)$, is the reference rate, a caplet with strike $\bar{L}$ will pay

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Caps and *floors* are derivative contracts that pay the option payoff every period. In other words, a cap will pay

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at specified times t_i, $i = 1, .., N$.

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Example. Lets assume that the LIBOR reference rate can be modeled as the spot-rate + 50 BP, and that the previous tree describes the spot-rate evolution. Assume 10M notional, $d = 180$, and $\bar{L} = 5.5\%$.

The cap pays

$$10,000,000 \times (0.055 + 0.005 - 0.055)\frac{180}{360} = 25,000$$

if rates go up (5.5%), and nothing if they go down (4.5%) after the first 6 months.

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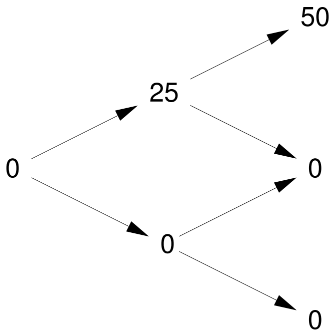
We have the following tree for the cap's payoffs:



It is now straightforward to compute the value of the cap at time zero. At time 1, up state it is

$$\frac{0.6489 \times 50 + (1 - 0.6489) \times 0}{1 + 0.055/2} + 25 = 56.57.$$

Similarly at time 0 we get

$$\frac{0.8024 \times 56.57 + (1 - 0.8024) \times 0}{1 + 0.05/2} = 44.27.$$

So the value is 44.27 at time 0.

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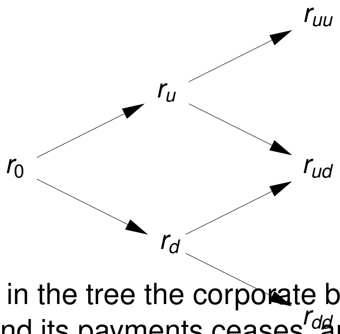
Valuing Corporate Callables

- Many of the corporate bonds we observe are callable.
- So let's augment the usual tree model to allow for the possibility of default.
- For simplicity, let's assume a constant hazard rate (probability of default).
- We will proceed by allowing for the possibility of default at each branch in the tree. In the case of default we recover X as before. The default probability is p.

As before, we have a standard tree for the default free spot rate:



but at each branch in the tree the corporate bond can default with probability p and its payments ceases, and we recover X.

Quoting Implied Volatility

Fixed income options are traded in the OTC market. The are often quoted in terms of *implied volatility*. Instead of using the standard Black & Scholes formula, the market convention is to use Black's (1976) model. The price of a cap in Black's model is

$$
\begin{aligned}
CAP_t(\sigma) &= \sum_{j=1}^{N} Caplet_j \\
&= \text{notional} \times \frac{d}{360} \sum_{j=1}^{N} B(T_j) \left[F(T_j) N(d_j) - \bar{L} N(d_j - \sigma \sqrt{T_j}) \right] \\
d_j &= \frac{\ln \frac{F(T_j)}{\bar{L}} + \sigma^2 T_j}{\sigma \sqrt{T_j}}
\end{aligned}
$$

where

- σ is the (implied) volatility of the cap
- $F(T_j)$ is the forward rate associated with a forward on a zero with maturity T_j delivered at time T_{j-1}.
- $B(T_j)$ is the price of a T_j maturity zero