No-Arbitrage Discount Functions

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Valuation

In baby finance, we compute the present values of cash flows received in 1,2,..,T years from now. The value of a security that pays a cash flow $C(t), t = 1, .., T$ is

$$V_0 = \sum_{t=1}^{T} C(t)d(t) \tag{1}$$

where $d(t)$ is a *discount function*. We discuss this in more detail below, but if interest rates are constant, then

$$d(t) = (1 + r)^{-t} \tag{2}$$

such that

$$V_0 = \sum_{t=1}^{T} \frac{C(t)}{(1 + r)^t} \tag{3}$$

HOWEVER this is not good enough!

We want a more GENERAL discount function :

- If we use equation (2) we are essentially discounting all future cash flows with the same rate (r)!
- We don't want that! It is too restrictive.
- In fixed income markets, we get a different rate of return depending on the *maturity* of the bond.
- We also need a discount fn that can take fractional arguments because we cannot restrict ourselves to securities that only pay interest in exactly even number of years from now

Let us instead assume that a semi-annual coupon bond pays its first coupon in 30 days from now. In this case, the first coupon is in 1/12=0.083333 years from now, the second is in 0.58333 years from now, the third in 1.08333 years from now and so on.

Lets label the n coupon dates by $t_1, t_2, ..., t_n$. The present value is

$$V_0 = \sum_{i=1}^{n} C(t_i)d(t_i). \tag{4}$$

In order to use this formula, we need to figure out what the discount function $d(t)$ is at the specific times $t_1 = 0.0833, t_2 = .5833$....

REMEMBER THIS FORMULA. IT REPLACES/GENERALIZES THE USUAL PRESENT VALUE FORMULA.

The Discount Function

The discount function, $d(t)$, denotes today's value of \$1 paid t years from now.
Suppose, for example, that $d(0.5) = 0.98$ then the value of one dollar paid six months from now is

$$0.98 \times 1 = 0.98,$$

while the value of \$4 paid six months from now is

$$0.98 \times 4 = 3.92.$$

Likewise the value of a \$104 paid in one year from now is (assume $d(1) = 0.95$),

$$104 \times 0.95 = 98.8$$

Example: A 8% semi-annual coupon with exactly one year to maturity pays 4 in 6 months and 104 in 12 months. It is thus worth

$$0.98 \times 4 + 0.95 \times 104 = 102.72.$$

It should be clear: If we know the discount function $d(t)$, we can price bonds...
...so how do we get it?

Computing the Discount Function from bond Prices

We will consider a simple enough method for computing the discount function. We do this using a real world example. Consider the following bond prices:

Table : Bond Prices on July 15 2008

CUSIP	MATURITY	CPN	NXT_CPN_DT	PX_DIRTY_ASK
912828EC0	8/15/2008	4.125	8/15/2008	101.9455701
912828EV8	2/15/2009	4.5	8/15/2008	103.2730082
912828CS7	8/15/2009	3.5	8/15/2008	102.8112981
912828DL1	2/15/2010	3.5	8/15/2008	103.3425481
9128276J6	8/15/2010	5.75	8/15/2008	109.0103022
9128276T4	2/15/2011	5	8/15/2008	108.0741758

- Note that these are all semi annual coupon bonds with February/August coupon and maturity payments.
- Assume for now that we can trade at the asking price.
- Initial date is July 15, 2008.
- Use 360 days a year convention.

Step 1: Finding $d(0.0833)$.

The first bond matures at 8/15/2008 with coupon rate 4.125% and thus has 30/360=0.0833 years to expiration. It pays 100+4.125/2 = 102.0625 on 8/15/2008.
It must be that

$$101.9455701 = d(0.0833)102.0625,$$

or

$$d(0.0833) = \frac{101.9455701}{102.0625} = 0.998854.$$

Thus, we have managed to figure out what $d(0.0833)$ is....

Step 2: Finding $d(0.5833)$

The next step is to find the value of a dollar paid on 2/15/2009.
This is in seven months from July 15, 2008.
To figure out what $d(0.5833)$ is, we consider the 2/15/2009 maturity bond.
The 2/15/2009 bond pays 4.5% semi annual coupon.
The first coupon payment is on 8/15/2008 when it pays 4.5/2=2.25. The present value of this coupon is $2.25 \times d(0.0883) = 2.25 \times 0.998854 = 2.247422242$.

The market value of the Feb 09 bond is 103.2730082 and it must satisfy

$$103.2730082 = 2.25 \times d(0.0833) + 102.25 \times d(0.5833)$$

Since $d(0.0833) = 0.998854$ we have that

$$103.2730082 = 2.247422242 + 102.25 \times d(0.5833)$$

or

$$d(0.5833) = \frac{103.2730082 - 2.247422242}{102.25} = 0.988025291.$$

The remaining ones...

We similarly find that

$$d(1.08333) = \frac{Price - .5 * coupon \times S}{100 + .5 \times coupon} \quad (5)$$

where S is the sum of all prior values of the discount function (in this case $S = 0.998854 + 0.988025$).
We get $d(1.0833) = 0.976258071$.

For $t = 1.5833$, we find

$$S = 0.998854 + 0.988025 + 0.976258 = 2.963137692$$

so that

$$d(1.5833) = \frac{103.3425481 - \frac{3.5}{2} \times 2.963137692}{100 + \frac{3.5}{2}} = 0.964688522$$

Similarly, we find
$d(2.0833) = 0.949869277$, $d(2.5833) = 0.93541402$, *etc.*
The next figure plots the entire discount function computed from both bids and asks on July 15, 2008.

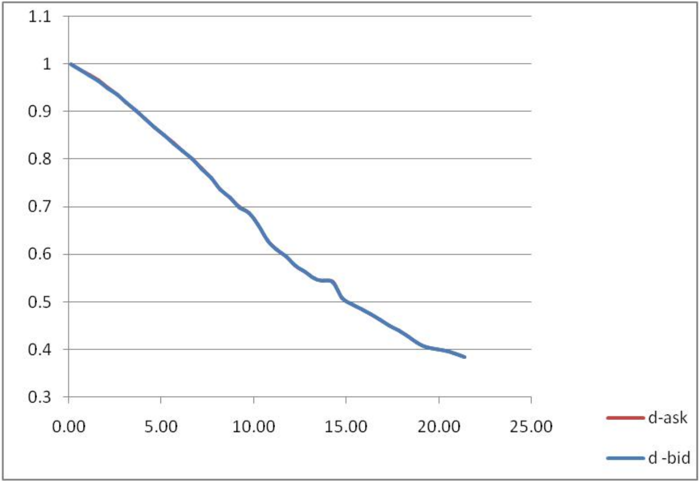

Figure : Discount function computed from bids and asks using FEB/AUG maturities on Jul 15, 2008

Comments on the discount function:

- Function computed from both asks and bids are visually indistinguishable
- Notable humps in the 15-17 yr range. Humps could be due to micro market structure effects. They COULD imply possible arbitrage opportunities (more on that later...)
- Notice that the 15-17 year range consists of old, off-the-run 30 years
- Curve flattens out at the end

Re-visiting zero coupons

Remember from last time: zero coupon bonds pay principal only.
Consider a zero coupon with maturity t and with \$1 principal. What is the current value?
It is just $d(t)$.
The discount function is identical to prices of zero coupon bonds if such exist (otherwise arb)
The *yield to maturity*, y_t of the zero coupon bond solves

$$d(t) = (1 + y(t))^{-t},$$

such that

$$y(t) = d(t)^{-\frac{1}{t}} - 1$$

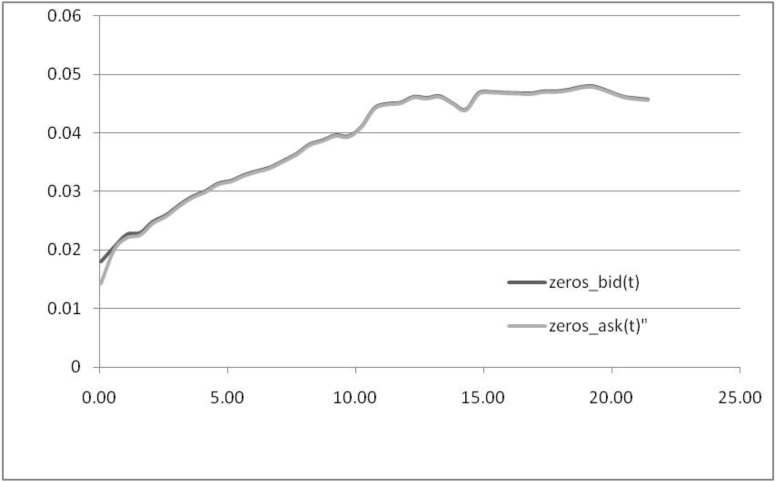

Figure : Zero coupon yield curve implied by bids and asks using FEB/AUG maturities on Jul 15, 2008

The spot rate function

Consider any point on the discount function, for example $d(0.5833)$. This is the value on 7/15/2008 of one dollar paid on 2/15/2009. This is the value *implied* by the prices of coupon bonds. The quantity

$$1/d(0.5833) - 1 = 1.1875\%$$

is the rate of return on the 7 month investment implied by the discount function.
The number

$$r(t) = 2(d(t)^{-\frac{1}{t2}} - 1)$$

is the *annualized spot-rate implied by the discount function.* The spot rate $r(t)$ is a semi-annually compounded yield curve associated with zero coupon claims.

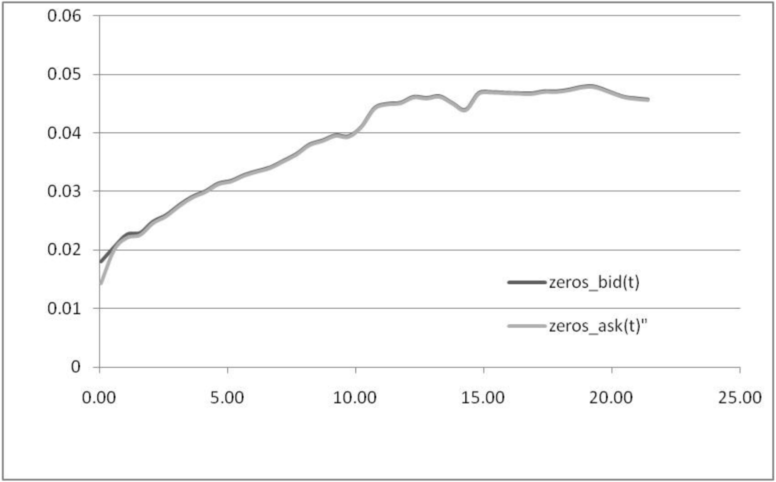

Figure : Spot rates implied by bids and asks using FEB/AUG maturities on Jul 15, 2008

Comments on the spot rate function:

- The spot rate function is in principle the same as the zero-coupon yield curve: It give the theoretical value of one dollar at some maturity t
- The previous spot rate curve is reasonably smooth in the 0-10 and 20-30 maturities, and relatively bumpy in the 10-20.
- It is a fairly typical zero coupon yield curve with an *upward slope*
- It is somewhat less common to find the *dip in the end*
- We will compare to zero coupon curves constructed using different methods later..
- We *could have* also computed a spot rate function using NOV/MAY maturities (see upcoming homework)

Shortcomings of the sequential method

- Does not work when there are no quotes for a particular maturity
- Requires all prices to be accurate. If not, $d(t)$ is not accurate eitherFor example, stale quotes for off-the-run 30 years may affect computations for the 10-20 year range.
- Yield curve looks strange with bumps and kinks. Could actually imply *negative forward* rates (see below) which again implies arbitrage.
- Almost certainly misprices NOV/MAY maturities.

Forward Rate Curve

Suppose there exists a forward contract such that we can enter into an agreement for a deliver of a T maturity zero coupon with a \$1 face in year $T - 1$.
Let $F(T)$ denote the forward price.
By entering into the forward contract, you receive the following cash flows (CF):

- At time 0: CF=0
- At time T-1: CF=-F (we pay the forward and receive the bond)
- At time T: CF=1 (bond pays principal)

The forward contract implies a cash outflow of $F(T)$ (the forward price) at time $T-1$ and gives \$ 1 at time T.
In other words, by entering the forward contract, the investor *locks in* a risk free return of

$$f(T) = 1/F(T) - 1.$$

We can show by an absence of arbitrage argument that

$$F(T) = \frac{d(T)}{d(T-1)}$$

implying that

$$f(T) = \frac{d(T-1)}{d(T)} - 1$$

is the forward rate.

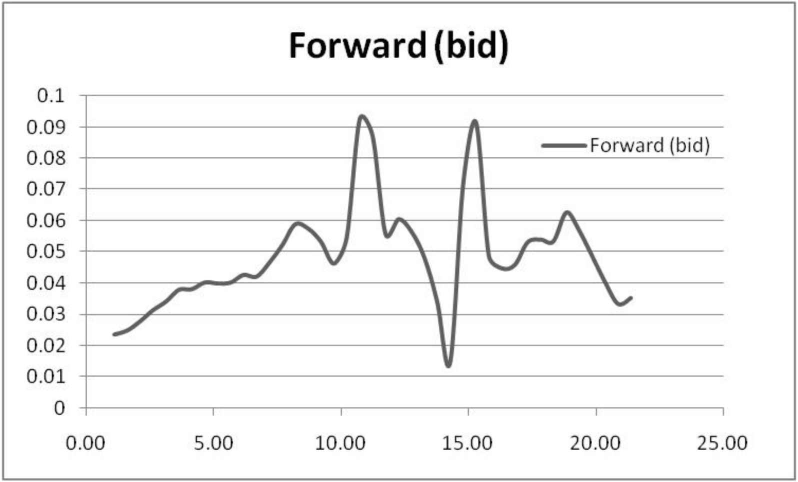

Figure : Forward rates implied by bids using FEB/AUG maturities on Jul 15, 2008

Features of the forward rate curve

- Many humps and bumps.
- Rates are positive
- Bumps are accentuated by bumps in the estimated zero curve
- Example: from Feb 2024-2025 forward rate is 1.8%. From Feb 2025-2026 we get 7.8%. Does this make sense?

Next time: Curve fitting allows us to estimate zero coupon curves that have less humps and bumps...