Forward Contracts

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Basic definition

A forward contract on some asset is an agreement today to purchase the asset at an agreed upon price (the forward price) today, for delivery some time in the future (settlement date).

Notation: (more detailed than Tuckman)

- $P(t,T)$: The time t price of a T maturity zero. Remember that $P(t,T)$ is the same as the discount function $d(t,T)$.
- $F(0,S,T) = F$: The time 0 forward price for a T maturity zero with settlement S $(S < T)$.

Cash flows from a forward contract

Assume initial date is time 0:

- At time 0 : No cash flows. Enter contract. Agree to pay $F(S, T)$.
- At time S : $-F$ (pay the forward price, receive the bond)
- At time T : Zero matures and we receive \$ 1.

Pricing a forward on a zero

Consider the following strategy at time 0:

- Buy the forward on a T maturity zero with settlement S.
- Sell one T zero short. Collect $P(0, T)$.
- Invest proceeds in the S zero.

We invest in the S zero for a total amount of $P(0, T)$.

The price of the S zero is $P(0, S)$, so the total number of bonds we buy is

$$n = \frac{P(0, T)}{P(0, S)}$$

such that $n \times P(0, S) = P(0, T)$ - the total proceeds from the short sale.

We now have the following cash flows:

At time 0: 0

At time S: - pay forward $-F(S, T)$
- collect \$ 1 coupon for every n S bonds.
Total cash flow is

$$\frac{P(0,T)}{P(0,S)} - F(S,T)$$

At time T: 0

Since the time S profit is riskless, it must be that

$$F(S,T) = \frac{P(0,T)}{P(0,S)} \tag{1}$$

is the arbitrage free price.

Forward rates

The forward contract is and investment F at time S yielding \$ 1 at time T. Therefore,

$$F(S,T) = \frac{1}{(1+f_{S:T})^{T-S}} \tag{2}$$

implicitly defines the forward rate, $f_{S:T}$, as the rate of return implicit in the forward price $F(S,T)$.

If $T - S$ (the maturity of the forward) is one year then we use the short-hand notation $f_{T-1:T} = f_T$ and the forward rate is

$$f_T = 1/F(T-1,T) - 1 \tag{3}$$

Coupon Bonds

Lets look at a coupon bond with a coupon payment $c/2$ taking place at some date T_c. Let's first assume that the coupon date is before the expiration of the forward. We assume for simplicity that the bond only pays coupon once before the forward expiration. Let $P_c(t)$ denote the time t (dirty) price of a maturity coupon bond. (Here, the maturity date of the coupon bond is not important and we omit is from the notation)

By entering into the forward contract, we receive the bond at time S but we do not receive the first coupon payment. Consider the following strategy at time 0:

- Buy the forward on the coupon bond.
- Sell the coupon bond short. Collect $P_c(0)$.
- Invest

$$\frac{c}{2}P(0, T_c)$$

  in the T_c maturity zero.
- Invest the remanding proceeds from short sale in S zeros. This gives us

$$n = \frac{P_c(0) - \frac{c}{2}P(0, T_c)}{P(0, S)}$$

  units of S zeros.

Cash flows:
At time 0: 0
At time T_c: Receive $c/2$. Use to pay coupon holders. zero cash flow.
At time S. Pay forward price,

$$-F$$

Use bond to cover short. Long position in S zeros pays n. The total cash flow is $n - F$.

So it must be that the arbitrage free forward price is

$$F = \frac{P_c(0) - \frac{c}{2}P(0, T_c)}{P(0, S)}. \tag{4}$$

Generalization:
As you may imagine, the price of a forward on a coupon bond with multiple future coupon payments at times t_i $i = 1, .., n$ is just

$$F = \frac{P_c(0) - \frac{c}{2}\sum_{i=1}^{n} P(0, T_i)}{P(0, S)} \tag{5}$$

Forwards vs Futures

Later: Futures contracts.
Note

- Forward and futures prices differ when interest rates are stochastic.
- Actual bond futures are typically not written on a specific bond. Rather, the issuer has the *option* to deliver any bond in a basket (called cheapest to deliver option). This make bond/note futures complicated.

Expectations Hypothesis

An old theory of forward rates suggests that they are expected future interest rates. Let f_T denote a T maturity forward rate

$$f_T = 1/F_T - 1$$

The EH suggests that

$$E(r_{T-1:T}) = f_T$$

I.e., the expected future short rate between $T-1$ and T equals the forward rate.

Risk-Neutral Expectations

In reality, the Expectations Hypothesis does not work. Rather, the theoretical relationship between the forward rates and the expected future short rate path contains a risk premium component

$$f_T = E(r_{T-1:T}) + \text{risk premium}$$

However, we often think of market implied expectations as something that would hold in a hypothetical world without risk premia. As such we simply define an expectation

$$E^Q(r_{T-1:T}) = f_T$$

where the E^Q notation denotes this market-implied expectation. It is most often called a *risk-neutral expectation*. The theory of risk-neutral expectations come back time and time again in theoretical finance.

The Relationship between Forward Rates and Zeros

Remember that $P_T = (1 + y_T)^{-T}$ where y_T is the YTM on the T year zero.

We have

$$\begin{aligned} 1 + f_T &= 1/F_T = \frac{P_{T-1}}{P_T} \\ &= \frac{(1 + y_{T-1})^{-(T-1)}}{(1 + y_T)^{-(T)}} \end{aligned}$$

or

$$1 + f_T = \frac{(1 + y_T)^T}{(1 + y_{T-1})^{(T-1)}} \qquad (6)$$

an equation often times found in basic finance textbooks.

We can now compute the zero coupon curve from a forward curve or vise versa using eqn. (6). Let's consider an example:

Suppose the forward rates are

T	f_T
1	0.5
2	0.75
3	1.1
4	1.5
5	2

By definition, we take $f_1 = y_1$ (the one-year forward rate is the same as the one-year zero YTM).

Solving for y_2 we find

$$(1 + y_2)^2 = (1 + 0.75/100) \times (1 + 0.5/100)$$

so $y_2 = 0.0062$ (62 bp).
Solving for y_3 we find

$$(1 + y_3)^3 = (1 + 1.1/100) \times (1 + 0.62/100)^2$$

so $y_3 = 0.078$. And so on.

We leave as an exercise to compute the rest of the zero curve.

Forward Rates Implicit in the Zero Curve

Bellow is a zero curve along with implied forward rates:

T	y_T	$d(T)$	f_T
1	5	0.9524	5.0000
2	4	0.9246	0.0301
3	3.5	0.9019	0.0251
4	3	0.8885	0.0151
5	3	0.8626	0.0300

You should verify the forward rates.

Let's see what happens if instead the 5th year zero is 2%. We get

T	y_T	$d(T)$	f_T
1	5	0.9524	5.0000
2	4	0.9246	0.0301
3	3.5	0.9019	0.0251
4	3	0.8885	0.0151
5	2	0.9057	-0.0190

Notice that the 5th year forward rate is now negative.

As it turns out, the drop from 3 to 2% from year 4 to 5 is “too large” to keep the forward rates from going negative.

Remember that any negative interest rate represents an arbitrage opportunity:

The MATTRESS technology: Borrow 1 billion at -1.9% interest between year 4 and 5. When year 5 is up, pay back 981 million. You keep the 19M.

For this to work you need a very big mattress. Also armed guards.

You also need to be able to carry out the arbitrage. In reality, you wouldn't because 1) arbitrages like this are too large to exist in (near) efficient markets, and 2) remember that zero coupons may not literarily be available for costless trading.

Forward Curve in Curve Fitting

We can parameterize directly a forward rate curve in order to do curve fitting. My favorite is to use a piecewise linear forward rate curve.

The spreadsheet 'discountFunctionsJanXX' contains a tab for curve fitting with a piecewise linear forward rate curve.

The 'knots' in the forward curve are the 'parameters' of the function. We can choose these in whichever way we want (presumably positive).

The relationship between forward and spot rates is described in Reading 42, p. 220. Eqn (1) defines the spot rate as the annually compounding yam:

$$p(T) = \frac{1}{[1 + r(T)]^T}$$

where of course T it bond maturity, in years.

Let $T*$ denote the start date of the forward contract (the same as S above). CFA denotes by $F(T^*, T)$ the forward contract we defined in the second slide as $F(S, T - S)$. So note that $T^* = T - S$ in their notation.

Eqn. (2), p. 221 in CFA is the same as my equation (1).

Eqn (3) in CFA is the same as our (2). Eqn. (4) in CFA is the same as our (6).

The rest of reading 42 up until section 3 should be straightforward. You should work through the examples.