Duration

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Bond Prices and Yields

- Consider a 2.625% coupon bond with maturity 3/15/2009. Suppose the bond is selling at with settlement 9/23/2008 at a price 100.534 implying a yield (to maturity) of 1.634%.
- We are interested in what would happen to the value of this bond if interest rates change.
- Suppose the ytm changes from 1.634 to 1.734 (a 10 basis point change). What is the impact on the price?

The price at $y = 1.734$ is 100.487 - a -0.047 dollar change.
Let $\triangle P$ and $\triangle y$ denote the change in the price and ytm, respectively.
We have $\triangle P = -0.047$ given $\triangle y = 0.1$. The relative change is

$$\frac{\triangle P}{\triangle y} = \frac{-0.047}{0.1} = -0.47$$

(the number is -0.4724 computed to four digits)

Let's see what happens if y changes by only 5 basis points. Then the new price is 100.510 and

$$\triangle P = 100.510 - 100.534 = -0.024.$$

The relative change is

$$\frac{\triangle P}{\triangle y} = \frac{-0.024}{0.05} = -0.4725$$

Note that this number is close to the relative change when $\triangle y = 0.1$.

Lets try even smaller $\triangle y$'s: We get

y(new)	P	$\triangle P$	$\triangle y$	$\triangle P/\triangle y$
1.634001	100.534	0.000	1E-06	-0.4726
1.635	100.534	0.000	0.001	-0.4726
1.644	100.529	-0.005	0.01	-0.4726

Note that as we change y very little, the ratio $\triangle P/\triangle y$ *approaches* the number -0.4726.

This number is the *derivative* of the price as a function of the ytm at the point $y = 1.634$. The derivative is denoted

$$\frac{dP}{dy}$$

We have found

$$\frac{dP}{dy} = -0.4726$$

at $y = 1.634$.
It is easy to demonstrate that a different price/ yield combination will imply a different derivative. For example if $y = 5.634$ then $dP/dy = -0.4592$.

DV01

The risk measure “DV01” measures the sensitivity of a bond’s price to a 0.01 change in the "interest rate," typically measured by the ytm of the bond.

DV01 gives the approximate absolute change in the value of the bond price in response to a 1 basis point change in the yield divided by 100,

$$DV01 = -\frac{\triangle P}{\triangle y}$$

with $\triangle y = 0.01$.

I.e, a change from 5% to 5.01%.

Dollar duration

The dollar duration of a bond is by definition

$$D_{\$} = -\frac{dP}{dy} \tag{1}$$

where importantly y is the YTM measured in percent.

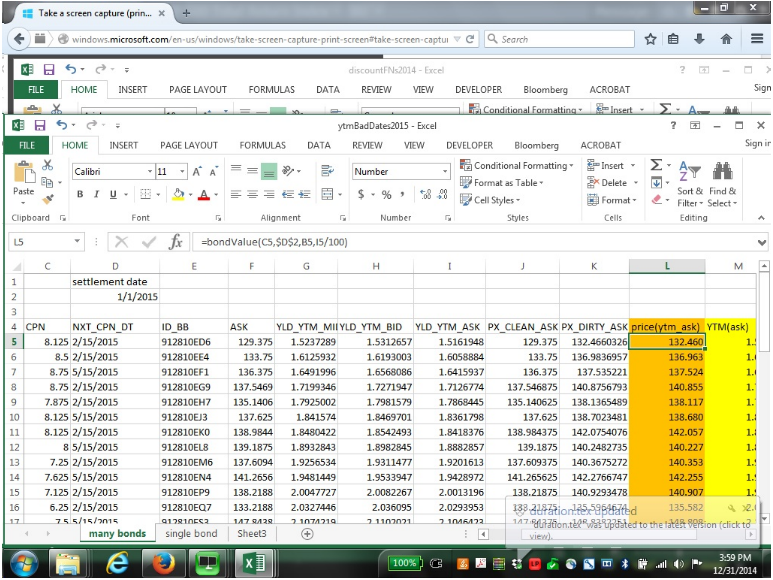
In our example $D_{\$}$ is just 0.4726.

To compute the dollar duration or DV01 in excel, we need a theoretical price (as function of YTM). The function "bondValue" in VB is included in the sheet "ytmBadDates2015.xlsm"

The arguments to "bondvalue" are

- coupon
- settlement date (excel date)
- expiration date (excel date)
- discount rate

When computed with discount rate = YTM, the value should by definition be equal to the market price.



	C	D	E	F	G	H	I	J	K	L	M
1		settlement date									
2		1/1/2015									
3											
4	CPN	NXT_CPN_DT	ID_BB	ASK	YLD_YTM_MI	YLD_YTM_BID	YLD_YTM_ASK	PX_CLEAN_ASK	PX_DIRTY_ASK	price(ytm_ask)	YTM(ask)
5	8.125	2/15/2015	912810ED6	129.375	1.5237289	1.5312657	1.5161948	129.375	132.4660326	132.460	1.
6	8.5	2/15/2015	912810EE4	133.75	1.6125932	1.6193003	1.6058884	133.75	136.9836957	136.963	1.
7	8.75	5/15/2015	912810EF1	136.375	1.6491996	1.6568086	1.6415937	136.375	137.535221	137.524	1.
8	8.75	2/15/2015	912810EG9	137.5469	1.7199346	1.7271947	1.7126774	137.546875	140.8756793	140.855	1.
9	7.875	2/15/2015	912810EH7	135.1406	1.7925002	1.7981579	1.7868445	135.140625	138.1365489	138.117	1.
10	8.125	5/15/2015	912810EJ3	137.625	1.841574	1.8469701	1.8361798	137.625	138.7023481	138.680	1.
11	8.125	2/15/2015	912810EK0	138.9844	1.8480422	1.8542493	1.8418376	138.984375	142.0754076	142.057	1.
12	8	5/15/2015	912810EL8	139.1875	1.8932843	1.8982845	1.8882857	139.1875	140.2482735	140.227	1.
13	7.25	2/15/2015	912810EM6	137.6094	1.9256534	1.9311477	1.9201613	137.609375	140.3675272	140.353	1.
14	7.625	5/15/2015	912810EN4	141.2656	1.9481449	1.9533947	1.9428972	141.265625	142.2766747	142.255	1.
15	7.125	2/15/2015	912810EP9	138.2188	2.0047727	2.0082267	2.0013196	138.21875	140.9293478	140.907	1.
16	6.25	2/15/2015	912810EQ7	133.2188	2.0327446	2.036095	2.0293953	133.21875	135.5964674	135.582	2.

Figure : Calling "bondValue" in "ytmBadDatesXXXX"

We can now compute the DV01 or dollar duration numerically by increasing the YTM by one basis point and dividing by 0.01 (multiply by 100):

In the excel spreadsheet we do this by calling "bondvalue" again using a single basis point higher YTM, take the difference, and divide by 0.01 (see next page)

Here we have the baseline bond value in column L. Then we call the "bondValue" again with a higher YTM. The result is in column P.

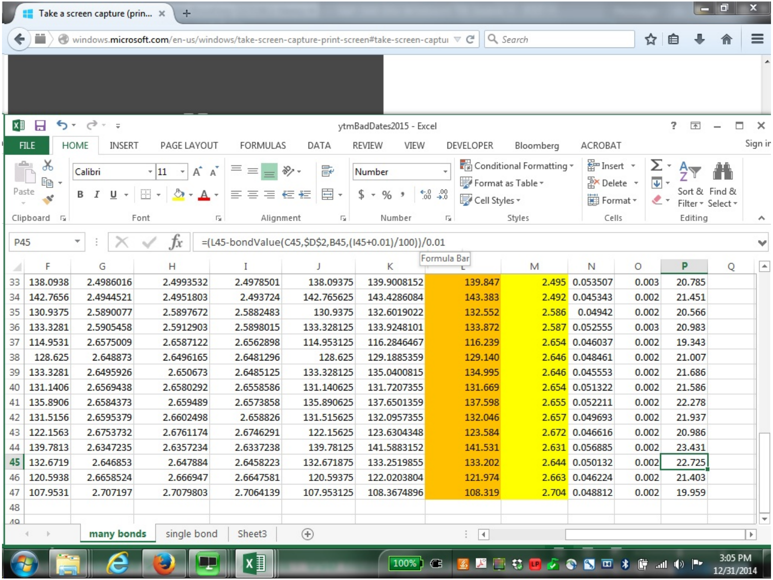


Figure : Numerical dollar duration compuation "ytmBadDatesXXXX"

Figure : YA for the May 41 maturity

Compared to Bloomberg

The "Risk" number in BB is the dollar duration/ DV01.

- BB gives 22.742
- We got 22.72
- If we compute it using $\triangle Y = 0.001$ we get 22.748

There are also slight differences in the way BB computes the present value vs. our spreadsheet.

A little calculus...

Remember that the value of the treasury with semi-annual coupons is

$$P = \sum_{i=1}^{n} \frac{C_i}{(1 + y/2)^{\frac{2t_i}{365}}}. \qquad (2)$$

(equation 4 in the "bondBasics" lecture).

We want to compute the derivative

$$\frac{dP}{dy}$$

analytically.

We get

$$\frac{dP}{dy} = \sum_{i=1}^{n} \frac{d}{dy} C_i (1 + y/2)^{-\frac{2t_i}{365}} \tag{3}$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \frac{2t_i}{365} C_i (1 + y/2)^{-\frac{2t_i}{365} - 1} \tag{4}$$

$$= -\frac{1}{(1 + y/2)} \sum_{i=1}^{n} \frac{t_i}{365} \frac{C_i}{(1 + y/2)^{-\frac{2t_i}{365}}} \tag{5}$$

$$= -\frac{1}{(1 + y/2)} \sum_{i=1}^{n} t_i^a PV_i \tag{6}$$

where

$$t_i^a = t_i/365 \tag{7}$$

is the time to cash-flow *i* in years, and

$$PV_i = \frac{C_i}{(1+y/2)^{-\frac{2t_i}{365}}} \tag{8}$$

is the present value of cash flow *i*.
Note that we can compute the dollar duration directly in VB if we please using equations (6) - (8).

VB implementation

```
Function dollarDuration(cpn, settle, expiration, y)
    nD = 182.5
    n = WorksheetFunction.CoupNum(settle, expiration, 2)
    firstCPNd = WorksheetFunction.CoupNcd(settle, expiration,
    s = firstCPNd - settle
    D = cpn / 2 * (1 + y / 2) ^ (-s / nD) * (s / nD)
    i = 1
    Do While i < n
        nextCPNd = nCPNdate(firstCPNd, i)
        s = nextCPNd - settle
        D = D + cpn / 2 * (1 + y / 2) ^ (-s / nD) * (s / nD)
        i = i + 1
    Loop
    D = D + 100 * (1 + y / 2) ^ (-s / nD) * (s / nD)
    dollarDuration = 0.5* D * 100 / (1 + y / 2)
End Function
```

Scaling

When calling the function in excel we get 227,420.

What's wrong?

The number is obviously scaled by 10,000 relative to the previous “correct" number of 22.74.

What happened?

Remember that we defined dollar duration to be the derivative wrt YTM measured in PERCENT. Equation (2) is a present value formula in which of course the discount rate is measured as a decimal number.

We thus need to scale the duration by 10000 to get the two numerical computation to agree. We do this by adding the code

```
dollarDuration = dollarDuration / 10000
```

to the end of the VB function.

Modified Duration

Define $\$D$ to be the Dollar Duration. The Modified Duration is defined

$$D_M = \frac{\$D}{P} \tag{9}$$

in most textbooks. P is the price of the bond.

It's easy to verify that dividing the dollar duration ("RISK") for the 4.75 May 41 Treasury note gives the wrong number. Using the BB numbers we get

$$\frac{22.742}{133.25} = 0.1707$$

which is obviously scaled by 1/100.

So the correct formula, given our definition of dollar duration, is

$$D_M = \$D \frac{100}{P} \tag{10}$$

which gives 17.067.

is defined

$$D = (1 + y/2)D_M \tag{11}$$

Using the BB numbers again we get

$$D = 17.067 \times (1 + 0.026458/2) = 17.293$$

Hedging

Suppose we have some position in a bond (or portfolio of bonds). For the sake of argument, suppose we guy 1M face value of the Nov 41, 4 3/8 maturity.

We wish to take a second position to offset potential interest rate moves. We can accomplish this by selling short a certain amount of a different bond. Let's consider the May 41 maturity.

On the next two pages you find the Bloomberg YA screen for these two bonds on Dec 31st, 2014.

Figure : YA for the May 41 maturity on Dec 31st, 2014

Figure : YA for the Nov 41 maturity on Dec 31st, 2014

A naive hedge

Let's just sell short 1M worth of face of the Nov maturity to see what happens.
We see that

- We pay $1,332,519 on Dec 31st for 1M face of the May maturity.
- We receive $1,083,674 in cash for shorting the Nov maturity.

Now let's see what happened to our position on Jan 2d, 2015:

Figure : YA for the May 41 maturity on Jan 2d, 2015

Figure : YA for the Nov 41 maturity on Jan 2d, 2015

We now unload the position at market prices. We get

- 1,347,569 for the May maturity
- We pay 1,096,746 to cover the short on the Nov maturity

In total we gained \$1,347,569 - \$1,332,519 = 15,050 on the long position and -1,096,746 + 1,083,674 = -13,072 on the short.

The hedge almost works and we get a gain of about 2K. This ideally should be zero. Let's see if we can do better....

Hedging with Dollar Duration

What messed up our hedge is that the DV01 on 1M of the May is higher than the DV01 on the Nov. (2,307 vs 1,995).

Meanwhile, the YTM on the two bonds went from 2.646 to 2.581 (down 6.5 BP) and 2.706 to 2.643 (down 6.3 BP), respectively.

Let F_a and F_b denote the face amount of each bond.
The following is the amount of face of a we need to hedge F_b:

$$F_a = \frac{-F_b DV01_b}{DV01_a} \tag{12}$$

So in our example,

$$F_a = -1M \frac{2274}{1995} = -1.1398M \tag{13}$$

By doing 1.1398 M of face of the Nov maturity we get a capital gain of

$$-1.1398 \times (1,096,746 - 1,083,674) = -14,900.$$

Remember that the gain on the May maturity was 15,050 so wet are off by only $150.

Moral of the story: The duration hedge works (almost perfectly).

When duration hedging does not work...

Consider the BB screens that follow on the next two pages.

Notice that for the 30 year bonds, the yields went down about 6-7 BP.

For the 1-2 year notes, yields did not change...

			BidPx / AskPx	AskYld	YlChg
31)	0¼	D15	99-31¾ / 100-00⅛	0.246	--
32)	2⅛	D15	101-25+ / 101-25⅞	0.287	-0.007
33)	0¼	D15	99-31¼ / 99-31¾	0.258	--
34)	0⅜	116	100-02⅜ / 100-02¾	0.291	+0.003
35)	2	116	101-24+ / 101-25⅛	0.328	-0.002
36)	0⅜	116	100-01⅞ / 100-02¼	0.309	-0.004
37)	4½	216	104-18⅛ / 104-19	0.355	+0.015
38)	9¼	216	109-28⅛ / 109-29⅝	0.297	-0.012
39)	0⅜	216	100-01 / 100-01+	0.333	-0.011
40)	2⅝	216	102-18⅝ / 102-19⅜	0.351	-0.006
41)	2⅛	216	102-00⅝ / 102-01¼	0.345	-0.003
42)	0¼	216	99-27+ / 99-28	0.359	-0.006
43)	0⅜	316	99-31⅞ / 100-00¼	0.368	+0.006
44)	0⅜	316	99-31⅜ / 99-31⅞	0.378	-0.006
45)	2⅜	316	102-14 / 102-15	0.367	-0.007
46)	2¼	316	102-09+ / 102-10¼	0.363	-0.006
47)	0¼	416	99-26⅝ / 99-27	0.373	+0.004
48)	2	416	102-01⅜ / 102-02¼	0.423	+0.002

			BidPx / AskPx	AskYld	YlChg
49)	2⅞	416	102-27⅛ / 102-28¾	0.417	+0.004
50)	0⅜	416	99-30 / 99-30⅝	0.408	+0.015
51)	0¼	516	99-24⅜ / 99-24⅞	0.414	+0.004
52)	5⅛	516	106-11 / 106-12⅛	0.413	+0.019
53)	7¼	516	109-05⅝ / 109-08¼	0.412	+0.010
54)	3¼	516	103-28⅝ / 103-29¾	0.434	-0.002
55)	1¾	516	101-25¼ / 101-26⅛	0.448	+0.006
56)	0⅜	516	99-29+ / 99-30¼	0.414	+0.003
57)	0½	616	100-02⅛ / 100-02⅝	0.443	+0.005
58)	0½	616	100-00⅝ / 100-01¼	0.474	+0.005
59)	3¼	616	104-02⅛ / 104-03+	0.472	-0.010
60)	1½	616	101-15+ / 101-16¼	0.481	+0.002
61)	3¼	716	104-08+ / 104-09¾	0.495	-0.007
62)	1½	716	101-16⅝ / 101-17+	0.510	--
63)	0⅝	716	100-06¼ / 100-06¾	0.486	--
64)	0½	716	100-00⅞ / 100-01⅜	0.472	+0.002
65)	4⅞	816	106-30⅝ / 106-31⅝	0.514	+0.007
66)	0⅝	816	100-04⅜ / 100-05	0.527	+0.002

Figure : 1-2 year notes on Jan 2, 2015

			BidPx	/	AskPx	AskYld	YlChg
31)	$2\frac{3}{4}$	N23	105-22	/	105-23	2.041	-0.059
32)	$2\frac{3}{8}$	824	102-10+	/	102-11	2.104	-0.061
33)	$7\frac{1}{2}$	N24	148-13+	/	148-15+	2.045	-0.060
34)	$7\frac{5}{8}$	225	150-14+	/	150-16+	2.064	-0.060
35)	$6\frac{7}{8}$	825	145-06	/	145-07+	2.098	-0.061
36)	6	226	137-25+	/	137-27	2.152	-0.061
37)	$6\frac{3}{4}$	826	146-23	/	146-25	2.170	-0.063
38)	$6\frac{1}{2}$	N26	144-23+	/	144-25+	2.190	-0.062
39)	$6\frac{5}{8}$	227	146-24+	/	146-26+	2.199	-0.063
40)	$6\frac{3}{8}$	827	145-12+	/	145-14	2.223	-0.064
41)	$6\frac{1}{8}$	N27	143-06	/	143-08	2.236	-0.065
42)	$5\frac{1}{2}$	828	137-16+	/	137-18	2.275	-0.063
43)	$5\frac{1}{4}$	N28	135-03	/	135-05	2.277	-0.066
44)	$5\frac{1}{4}$	229	135-15+	/	135-17	2.289	-0.066
45)	$6\frac{1}{8}$	829	147-10	/	147-12	2.291	-0.067
46)	$6\frac{1}{4}$	530	150-18+	/	150-20	2.314	-0.068
47)	$5\frac{3}{8}$	231	140-18+	/	140-19+	2.336	-0.068
48)	$4\frac{1}{2}$	236	134-27+	/	134-29	2.386	-0.070

			BidPx	/	AskPx	AskYld	YlChg
59)	$3\frac{7}{8}$	840	123-15	/	123-16+	2.610	-0.065
60)	$4\frac{1}{4}$	N40	130-28+	/	130-30	2.599	-0.065
61)	$4\frac{3}{4}$	241	141-08+	/	141-10	2.569	-0.065
62)	$4\frac{3}{8}$	541	134-05	/	134-06+	2.579	-0.067
63)	$3\frac{3}{4}$	841	121-30+	/	122-00	2.599	-0.065
64)	$3\frac{1}{8}$	N41	109-07+	/	109-09	2.640	-0.066
65)	$3\frac{1}{8}$	242	108-30	/	108-31+	2.658	-0.064
66)	3	542	106-13+	/	106-14+	2.666	-0.067
67)	$2\frac{3}{4}$	842	101-07	/	101-08	2.686	-0.065
68)	$2\frac{3}{4}$	N42	101-05	/	101-06	2.689	-0.066
69)	$3\frac{1}{8}$	243	108-21+	/	108-22+	2.682	-0.065
70)	$2\frac{7}{8}$	543	103-17+	/	103-19	2.693	-0.065
71)	$3\frac{5}{8}$	843	119-00	/	119-01+	2.669	-0.065
72)	$3\frac{3}{4}$	N43	121-20	/	121-21+	2.668	-0.064
73)	$3\frac{5}{8}$	244	119-03	/	119-04+	2.674	-0.064
74)	$3\frac{3}{8}$	544	113-30+	/	113-31+	2.684	-0.064
75)	$3\frac{1}{8}$	844	108-29	/	108-30	2.685	-0.063
76)	3	N44	106-11+	/	106-12	2.688	-0.064

Figure : 30 year notes on Jan 2, 2015

This tells you that if we had tried to hedge the position in the May 41 with a position in any of the 1-2 year bonds, the hedge would have been useless.

This is so because the short end of the yield curve did not move. Only the long end did.

Consequently:

- Duration is a useful measure of risk only if you believe that the yield curve shifts in parallel all the time (not true)
- To hedge a single bond position, you need a bond (or other instrument) with very similar characteristics (i.e, maturity). I.e, you need "apples to hedge apples"
- It is rather useless to think of duration as representing an accurate measure of interest rate risk. Generally, the short end of the curve moves around a lot more than the long end, meaning that duration overstates the risk of long-maturity bonds relative to short maturity bonds,
- Even if two portfolios have the same duration, they may exhibit different interest rate sensitivity

Key Rate Duration

The idea behind key rate duration is that, as we have seen, different segments of the yield curve will move around independently.

Key Rate Duration seeks to measure the sensitivity of a portfolio with respect to a local shift in the yield curve. This is most often done by measuring the sensitivity of a bond with respect to a change in a local segment of *the zero-coupon yield curve*.

Check out this youtube video for a stylized simpilied example:
`https://www.youtube.com/watch?v=nQ4nbF0rfUA`
Note that what he refers to as the spot rate curve is the zero-coupon yield curve (continuously compounding).

Some more math...

Let's try to understand (mathematically) why duration hedging work

The relationship between the dollar duration, the change in the yield, and the change in the bond price can be expressed approximately as

$$\triangle P \approx -\$D\triangle y \tag{14}$$

We can think of $\triangle P$ as representing the change in the (dirty) bond price from day t to $t+1$.

Suppose we have a portfolio with n_i number of bond i (so $n100$ is the face amount of bond i). The portfolio value is

$$V = \sum n_i P_i \tag{15}$$

We then have that the change in the portfolio value is

$$dV = \sum_i n_i \triangle P_i = -\sum_i \$D_i dy_i \tag{16}$$

To derive the hedging formula (12), assume that the yield on bond *a* change by as much as the yield on bond *b* ($dy_a = dy_b = dy$). We need

$$0 = n_a \$D_a dy + n_b \$D_b dy$$

which gives

$$n_a = -n_b \frac{\$D_b}{\$D_a} \tag{17}$$

or equation (12).

Convexity

Equation (14) is an example of a First Order Taylor expansion. You can guess that there exists a second (and third, and....) Taylor expansion which is more accurate:

$$\triangle P \approx \frac{dP}{dy}\triangle y + \frac{1}{2}\frac{d^2P}{dy^2}\triangle y^2 \tag{18}$$

where $\frac{d^2P}{dy^2}$ is the second order derivative of the price wrt the YTM.

The convexity of a bond is usually defined as

$$C = \frac{1}{P}\frac{d^2P}{dy^2} \tag{19}$$

Note that since the second order derivative is the same as the derivative of the first order derivative, we can think about convexity as measuring how much the durations change as the yield changes.

The convexities of the May and Nov 41 Notes are stated on the BB YA screens to be 3.826 and 4.331 respectively.

Let's consider constructing a convexity neutral portfolio. We need to solve

$$0 = n_a C_a + n_b C_b \tag{20}$$

in addition to the equation for the duration neutrality.

We now have two equations but only one unknown, n_b, which we will therefore not be able to solve. We would need one additional bond to construct a convexity and duration neutral portfolio.