Black-Derman-Toy

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Trees that fit the current term structure

Consider a world in which you observed the following zero coupon bond ytm's:

maturity	1	2	3	4	5
ytm	0.1	0.11	0.12	0.12.5	0.13
price	0.901	0.812	0.712	0.624	0.543

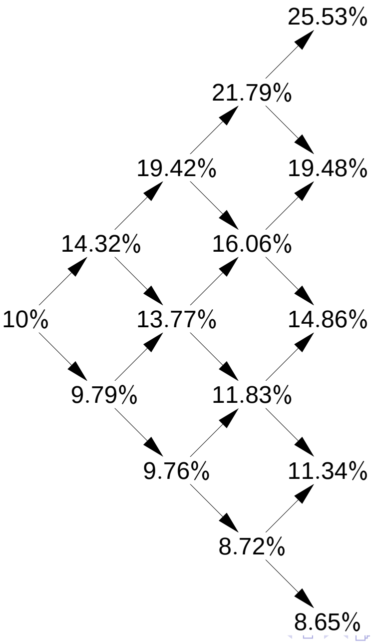
How can we build a tree such that it prices these zero coupon bonds correctly?

Yes, and there are several methods available for constructing such trees:

- Ho-Lee
- Hull White
- Black-Derman-Toy
- Black-Karasinsky

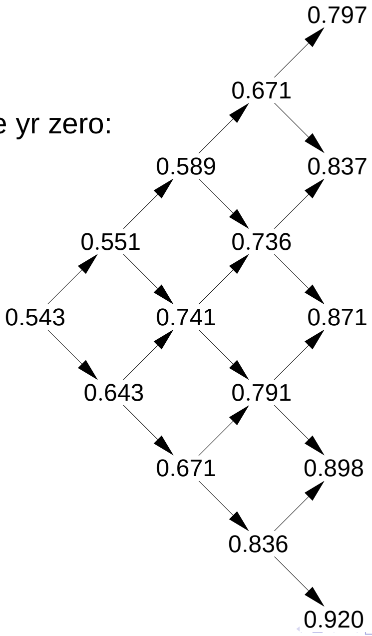
These models make different assumptions about the behavior of interest rates. We will consider first and example from Black-Derman-Toy.

BDT tree:

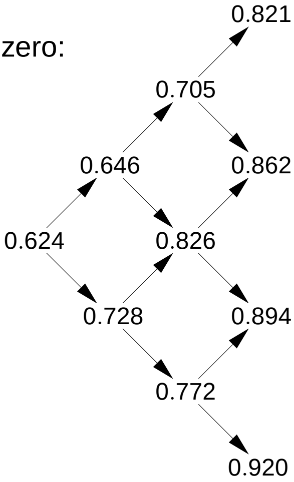

25.53%
21.79%
19.42%
19.48%
14.32%
16.06%
10%
13.77%
14.86%
9.79%
11.83%
9.76%
11.34%
8.72%
8.65%

How do we know that this tree is consistent with the current term structure?
Let's price a 5 year zero in the tree. It is...

Price tree for the five yr zero:



Similarly, the four yr zero:



Thus, we verify the prices of both the 5 and 4 year zeros. We can do the same exercise for the 3 and 2 of course....

Properties of the BDT model

- Model has 50/50 risk neutral probabilities of up/down moves
- The short term interest rate is *log-normally* distributed
- Rates are always positive
- Model assumes that you can estimate future short rate volatility. We need a volatility model to make such forecasts (i.e., GARCH)

Volatility

Lets look at the ratio of the up/down states at some random time/state, for example the up/up/up and up/up/down, in which case the rates are 25.53 and 19.48. The ratio is

$$\frac{25.53}{19.48} = 1.31$$

Going one step down, we have

$$\frac{19.48}{14.86} = 1.31$$

... further ...

$$\frac{14.86}{11.34} = 1.31$$

...in other words.. the ratio of the up/down states are the same for each state.
At time step 4 we find the ratio to be 1.35, at time 3 we find 1.41, at time 2 we find 1.46.
A measure of volatility in the BDT model is

$$\sigma(t) = \frac{1}{2} \ln r^u(t)/r^d(t)$$

Since the ratio of the up/down rates are the same irrespective of the level of interest rates, $\sigma(t)$ does not depend on the *level* of interest rates, as in CIR.

$\sigma(t)$ is however *maturity specific*. This means that the BDT model allows for the volatility of future short rates to depend on time.

Example: Suppose we were to believe that future short rates will be less volatile than today, then we could specify a decreasing volatility structure, for example

$$\sigma(t) = [0.2, 0.16, 0.14, 0.13, 0.12...]$$

would suggest that we expect volatility to decrease from its current level of 200 basis point/ year to 120 basis point in the next 5 years...
On the other hand, we could specify

$$\sigma(t) = [0.05, 0.07, 0.09, 0.1, 0.11...]$$

to suggest an increasing volatility structure.
The downward (upward) sloping volatility scheme would be practically relevant when financial markets are very volatile (not volatile).

Volatility in the BDT model

The volatility measure is to be interpreted in relative terms. For example, if the current interest rate is 10%, a volatility of 0.19 means that a likely move in interest rates is 19% up or down, so

$$r^u \approx 1.19 * 0.1 = 0.119 + \text{expected change}$$

and

$$r^u \approx 0.1/1.19 = 0.084 + \text{expected change}$$

In the previous lecture we saw that the BDT example tree specified a move from 10% either 14.32% or 9.78% in the first period. We also expect interest rates to change from 10 to 12% in the first period.

Therefore, a volatility of 0.19% suggests that interest rates should change to approximately 0.119+.02 = 0.139 and 0.1.

These numbers compare to the actual r^u and r^d in the first period of 0.1432 and 0.0978.

Measuring and forecasting volatility

Lets look at historical short rates. We want to measure the volatility of $\ln r(t) - \ln r(t-1)$ - the logarithmic change in interest rates we observe over time.
Note the following facts:

- Volatility of $\ln r(t)/r(t-1)$ for 6M T-Bill rates from 1959 to Oct 28, 2008 is 0.157.
- Volatility of $\ln r(t)/r(t-1)$ for 6M T-Bill rates from Jul 28, 2008 to Oct 28, 2008 is 0.43.

Lets look at a graph...

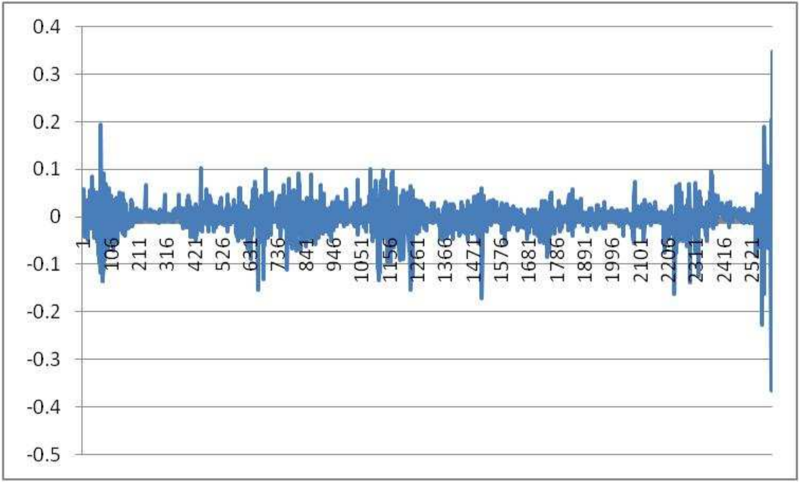


Figure: Relative log changes in 6M T-bill rates, Jan,1959-Oct,2008

Facts about volatility:

- It is well known that volatility changes randomly over time.
- Volatility is mean-reverting. It reverts slowly.
- Recently (prior to Oct 28, 08), financial markets have experienced extreme levels of volatility.
- Volatility can be forecasted using statistical models similar to the AR(1) models for the interest rate

A Volatility Forecasting Equation

We will simply assume that expected volatility in period t is

$$EV_t = EV_{t-1} + \kappa_V(EV - EV_{t-1})$$

where κ_V is the speed of mean reversion for volatility. There are several estimates of κ_V in the literature. Eraker (2001) estimates

$$\kappa_V = 0.02$$

Using weekly T-Bill rates. This corresponds to

$$\frac{52}{2} \times 0.02 = 0.52$$

at a six month frequency.

We now take the most recent volatility measurement 0.43 and computes the following forecasts:
6 month forecast:

$$EV_6 = 0.43 + 0.52 \times (0.157 - 0.43) = 0.285$$

The 12 month forecast is

$$EV_{12} = 0.285 + 0.52 \times (0.157 - 0.285) = 0.22$$

and we get $EV_{18} = 0.187$, $EV_{24} = 0.172$, etc.

Fitting the Yield Curve

I refitted a Nelson-Siegel yield curve function on 10-28-2008. The fitted parameters were $b_0 = -0.259, b_1 = 0.261, b_2 = 0.582$ and $\lambda = 0.0382$.

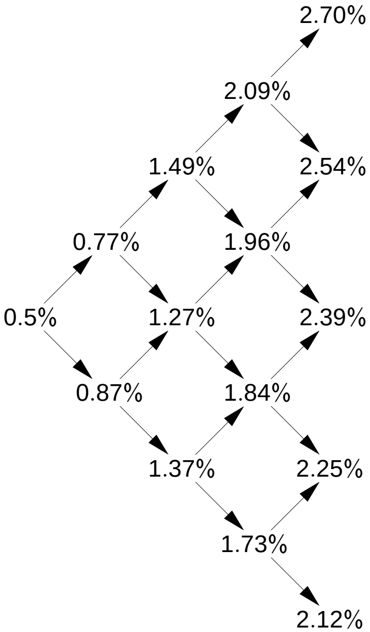
The pricing errors were quite large, driven by high prices on long dated (30 year) bonds (recent flight to quality).

We can now build the BDT tree.

It is implemented in the excel spreadsheet "bigBDT.xls."

- The tree itself was built using the Matlab "Financial Derivatives Toolbox" using the function "bdttree"
- Built using 6 month intervals
- Control: We will use the tree to price coupon bonds and also an option on this bond
- First time step 10-28-2008 to 11-15-2008 - 17 days
- Remaining time-steps are 6 months.
- Valuation in the tree assumes equal number of days in each six month period. No weekend delayed coupons.

First few branches:



Pricing the 9%, 11/15/2018 coupon bond

We will price this bond just to see that the tree is in fact reasonably built. If it is "correct" it will produce a price close to the theoretical price obtained through the curve fitting exercise.

- Real price is 142.2
- Curve-fitted theoretical price is 144.
- BTD tree price: 142.5

Reasons for difference between theoretical prices

Difference is 1.5 cents. It could be due to

- Numerical rounding errors
- Bad dates
- Price from curve fitting obtained by measuring exactly the number of days to future coupon payments

Pricing European Options on 9 of Nov, 2018

Suppose we have a call option to buy the Nov 2018 with strike 120 and maturity 4.5 years (May 14, 2013 - before the coupon payment on the 15th)

The spreadsheet reveals that the price today of such an option should be $ 0.5811.

The corresponding put value is $0.42.

Probability distributions of future spot rates

The probability of the first branch (11/15/08) is 1/2 and 1/2. The

upper and lower branch at time 2 (5/15/09) have probability $0.5 \times 0.5 = 0.25$.

There are two ways to get to the middle branch on 5/15/09: You can go up, then down or down then up.
The combined probability is $0.5 * 0.5 + 0.5 * 0.5 = 0.5$.

The spreadsheet has a sheet, "probabilities," giving the probability of hitting each possible future state.

We can combine this with the associated spot rate levels to plot the probability density of future interest rates.

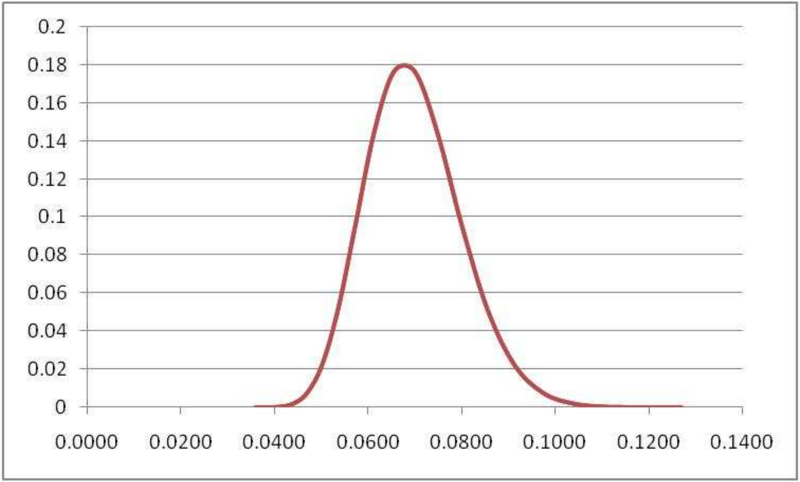


Figure: Probability density of the spot rate 10 year from now in the BDT model.

Minor concluding remarks

First time step: Need to include a time step to account for the first 17 days before the first coupon payment. Subsequent time steps are in 6 month time intervals.

Can do finer and coarser time steps: Ideally, we'd build a tree with higher frequency time steps over an initial period, then more course time steps.

Ex: Build a tree where the first year is in daily steps, second is in monthly, and subsequent time steps are in quarterly increments.