MANAGEMENT SCIENCE
Vol. 54, No. 12, December 2008, pp. 2068–2080
ISSN 0025-1909 | EISSN 1526-5501 | 08 | 5412 | 2068

DOI 10.1287/mnsc.1070.0796

Affine General Equilibrium Models

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No-arbitrage models are extremely flexible modelling tools but often lack economic motivation. This paper describes an equilibrium consumption-based CAPM framework based on Epstein-Zin preferences, which produces analytic pricing formulas for stocks and bonds under the assumption that macro growth rates follow affine processes. This allows the construction of equilibrium pricing formulas while maintaining the same flexibility of state dynamics as in no-arbitrage models. In demonstrating the approach, the paper presents a model that incorporates inflation such that asset prices are nominal. The model takes advantage of the possibility of non-Gaussian shocks and model macroeconomic uncertainty as a jump-diffusion process. This leads to endogenous stock market crashes as stock prices drop to reflect a higher expected rate of return in response to sudden increases in risk. The nominal yield curve in this model has a positive slope if expected inflation growth negatively impacts real growth. This model also produces asset prices that are consistent with observed data, including a substantial equity premium at moderate levels of risk aversion.

Key words: finance; investment; asset pricing; probability; diffusion; stochastic model applications
History: Accepted by David A. Hsieh, finance; received August 14, 2006. This paper was with the author 5 months for 2 revisions. Published online in *Articles in Advance* September 11, 2008.

1. Introduction

Traditional models of financial market equilibrium, such as the Sharpe-Linter-Mossin CAPM and Lucas' consumption C-CAPM, are well known to fail in explaining essential stylized facts of asset market data. The C-CAPM cannot generate equity returns that on average exceed the risk-free rate unless risk aversion is unreasonably high (the equity premium puzzle), and even with a high-risk aversion, the model produces a risk-free rate that far exceeds those observed historically (risk-free rate puzzle). Clearly, the simple constant relative risk aversion (CRRA) model offers a too-simplistic notion of equilibrium to generate realistic pricing implications.

Although equilibrium models based on time-separable preferences have been uniformly rejected, the recent literature based on Long-Run-Risk has shown promise in explaining a number of asset-pricing puzzles. Noticeably, Bansal and Yaron (2004) derive a model in which one or two persistent long-run factors impact the first and second conditional moments of consumption growth. They show that a recursive utility function, rather than time-separable CRRA preferences, leads to high-equity premiums. The Epstein-Zin preference structure, by separating elasticity of substitution from risk aversion, allows for equilibria in which high-equity returns coincide with low-risk free rates.

The primary goal of this paper is to describe a convenient framework for pricing assets using a general equilibrium, Epstein-Zin based model economy. This framework is an extension of the Bansal and Yaron model economy in which all shocks in the economy are assumed to be normally distributed. Rather, equilibrium prices in this paper are derived under the assumption that exogenous state variables follow affine processes observed at discrete time intervals. The affine framework offers a very wide range of possible statistical behavior for the exogenous shocks in the model. Specifically, the affine framework allows for large shocks (modelled by Poisson-jump processes) as well as stochastic volatility. Another distinct advantage of affine processes is that they may be constrained to be positive. This is useful for modelling quantities that inherently should be positive, such as volatility. This contrasts with conditionally normally distributed volatility processes currently being used in the long-run-risk literature.

This paper presents a new model designed to illustrate some of the asset pricing implications of the affine equilibrium framework. This model is a generalization of the model in Bansal and Yaron (2004) to an economy in which the volatility of the consumption growth rate exhibits large exogenous shocks (jumps) and where inflation contains a random component. Because the price/dividend ratio in this model is an exponential-linear function of the volatility, a large positive shock in volatility leads to a large negative shock in the stock price. This is akin to an endogenously generated market crash. The second effect of volatility jumps is that the risk premia associated with volatility sensitive assets, such as stocks, are substantially larger. In our example model, the presence of volatility jumps increases the equity premium by as much as one percent per annum relative to a model that has no volatility jumps.

The effects of introducing inflation are as follows: If inflation is neutral, shocks to expected inflation and inflation volatility are not priced, and the nominal equity premium and yield curve equal their real counterparts plus inflation. When expected inflation impacts the real economy, it acts as an additional long-run risk variable. This positively affects the equity premium. As a consequence, even smaller risk-aversion values are needed to generate a premium consistent with that observed in the United States. Another interesting implication is that nonneutral inflation implies a proportionally higher premium for long maturity bonds. This has the consequence that the model produces a positively sloping nominal yield curve. The positively sloping yield curve in our model obtains even though the risk aversion and elasticity of intertemporal substitution parameters are small.

As equilibrium models have proved too restrictive, models based on no arbitrage have largely dominated the asset pricing literature. This is true for stock market valuation, where, for example, the widely used Fama-French three-factor model can be interpreted in the framework of the no-arbitrage theory of Ross (1976), in yield curve and credit risk modelling, where models by Vasicek (1977), Duffie and Kan (1996), and Dai and Singleton (2000) are typical examples, as well as derivatives pricing (i.e., Heston 1993, Bates 1996, Duffie et al. 2000). On one hand, no-arbitrage theory provides a convenient tool for modelling because it imposes almost no restrictions on the statistical behavior of asset prices. On the other hand, no arbitrage models offer limited insight into the behavior of financial market participants. For example, although empirical studies of no-arbitrage pricing models typically produce estimates of market prices of risk associated with exogenous shocks, which by assumption impact prices, there is no unified way to establish whether such estimates are consistent with equilibrium pricing or are even economically justifiable.

The use of affine processes in connection with Epstein-Zin preferences leads to notable differences in equilibrium prices relative to equilibrium models based on Cox et al. (1985). In their equilibrium framework, Cox et al. consider a time-separable utility $U = \int_t^T u(C_s, Y_s)\, ds$ defined over consumption C and exogenous state variables Y. In the special case that consumers derive utility from consumption only, the state variables Y are *not priced* in equilibrium unless their exogenous shocks are contemporaneously correlated with consumption. Because it is hard to motivate why variables such as expected growth rates or growth rate volatility, should enter directly into the utility function, most papers that have studied equilibrium models of the style of Cox et al. have assumed that state variables in some way correlate contemporaneously with consumption. For example, in Cox et al. (1985), equilibrium interest rates are derived in an economy in which consumption depends explicitly on exogenous "production processes." In contrast, long-run risk models generally assign positive market prices of risk to exogenous shocks as long as those shocks have long-run implications for aggregate consumption and even if these shocks do not impact current consumption.

There are a number of recent papers that study the link between macroeconomic and term structure dynamics.[1] A typical approach in this literature is to specify an exogenous affine model for the macroeconomy as well as latent factors and then to assume that the short rate process is a linear function of these exogenous processes. In recent work, Piazzesi and Schneider (2006) study an equilibrium term structure model based on recursive preferences and a negative long-run relation between expected inflation and expected real growth.[2] The GE modelling framework presented in this paper can possibly be construed as one that imposes parametric equilibrium constraints on such no-arbitrage models. The equilibrium constraints are testable and possibly imply quite sharp restrictions on the yield curve data. The approach outlined here also clearly addresses the issue of which macroeconomic variables to use in yield curve modelling: all variables, observable or not, that affect dynamics of real consumption growth and inflation matter for the term structure. The affine general equilibrium framework therefore offers particular guidelines as to the selection of candidate variables in constructing yield curved models.

This paper is related to the extant literature in a number of ways. The equilibrium is constructed using an Epstein and Zin (1989) and Weil (1989) preference structure, which again is based on the recursive preference class of Kreps and Porteus (1978). We follow Campbell and Shiller (1987, 1988), Bansal and Yaron (2004), and Bansal et al. (2006) in computing a linearized pricing kernel to facilitate the construction of an affine pricing kernel. Bansal and Yaron (2004) demonstrate that their model, in which expected aggregate consumption and volatility follow first-order autoregressive (AR(1)) processes, leads to a resolution of the equity premium and short rate

[1] An incomplete list includes Evans and Marshall (1998), Ang and Piazzesi (2003), Ang et al. (2005), Piazzesi and Schneider (2006), Diebold et al. (2005), Duffee (2006), and Bikbov and Chernov (2005).

[2] This feature of their model is similar to the one considered here. The two papers were completed independently.

puzzles. A number of papers building on the Bansal and Yaron model demonstrate that long-run risk models may explain a number of asset pricing "puzzles," including the cross-section of equity returns (Bansal et al. 2004, Kiku 2006), foreign exchange (Colacito and Croce 2006, Bansal and Shaliastovich 2006), and options pricing (Benzoni et al. 2005, Eraker and Shaliastovich 2007). The use of continuous time "affine processes" has a long tradition in finance, and key contributions include studies by Black and Scholes (1972), Cox et al. (1985), Duffie and Kan (1996), and notably Duffie et al. (2000), who outline a very general class on which results in this paper are partly based.

The remainder of this paper is organized as follows. The next section outlines assumptions and derives equilibrium prices for stocks and bonds. Section 3 presents the example model of nominal prices and discusses various properties of the equilibrium, and §4 concludes.

2. Equilibrium Prices

We assume an economy in which a representative agent receives an endowment consumption stream C_t for consumption at discrete times $t = 1, 2, \ldots$. The agent has Epstein-Zin utility

$$U_t = [(1-\delta)C_t^{(1-\gamma)/\theta} + \delta(E_t U_{t+1}^{1-\gamma})^{1/\theta}]^{\theta/(1-\gamma)}, \qquad (1)$$

where δ is the agents' subjective discount rate, ψ is his elasticity of intertemporal substitution, and γ determines preference for intertemporal resolution of uncertainty (risk aversion). The parameter θ is defined as

$$\theta = \frac{1-\gamma}{1-1/\psi}. \qquad (2)$$

The defining feature of Epstein-Zin preferences is the recursive structure in which U_t depends on future expected utility U_{t+1}. This is a key ingredient in generating dynamic effects in long-run risk models. Intuitively, even if a shock to expected growth or volatility of dividends does not have an immediate effect on current consumption or dividends, if the shock has an impact on *future* consumption, it will impact the expected utility term $E_t U_{t+1}$ and thus impact the current utility U_t. This has implications for financial prices unless the utility function degenerates to a power utility (CRRA) utility, which happens when $\psi = 1/\gamma$.

The first-order condition for optimal utility produces the Euler equation

$$\begin{aligned} 1 &= E_t(\delta^\theta G_{t+1}^{-\theta/\psi} R_{a,\,t+1}^{-(1-\theta)} R_{i,\,t+1}) \\ &= E_t \exp\left(\theta \ln\delta - \frac{\theta}{\psi} g_{t+1} - (1-\theta) r_{a,\,t+1} + r_{i,\,t+1}\right), \end{aligned} \qquad (3)$$

where $G_{t+1} = C_{t+1}/C_t$ is the consumption growth rate, $R_{a,\,t+1}$ is the return on an asset that pays dividend equal to aggregate consumption, and $R_{i,\,t+1}$ is the return on an arbitrary asset with index i. Epstein and Zin interpret $R_{a,\,t}$ to be the return on the market portfolio.

2.1. Dynamics of Exogenous State Variables

We assume that there are n marketwide state variables X_t that are assumed to follow an affine process. This implies that expected values of exponentiated linear functions of these state variables can be written

$$\begin{aligned} \phi(s, u, x) &=: E(\exp(uX_{t+s}) \mid X_t) \\ &= \exp(\alpha(s, u) + \beta(s, u)' X_t) \end{aligned} \qquad (4)$$

for some u either real or complex. We refer to ϕ as the generating function. Appendix A gives a recipe for computing the functions α and β under the assumption that X_t follows an affine jump-diffusion process. Any process that satisfies (4) is termed an affine process.

There are two immediate examples that characterize the class of processes that have exponential linear (or affine)-generating functions, as in (4). The first class are Gaussian vector autoregressions (VARs). In other words, if

$$X_t = \mu + AX_{t-1} + \Sigma\epsilon_t,$$

the generating function is $\phi(1, u, X_t) = \exp(u'(\mu + AX_t) + \frac{1}{2}u'\Sigma u)$. Our model would coincide with that of Campbell (1993) if X_t follows a Gaussian VAR.

A second important class of processes that produce exponential affine-generating functions is affine jump-diffusion processes. An affine jump-diffusion process is a process $X_t \in \mathcal{D}$ for some $\mathcal{D} \subseteq \mathbb{R}^n$ described by

$$\mathrm{d}X_t = \mu(X_t)dt + \Sigma(X_t)dW_t + \xi_t dN_t, \qquad (5)$$

where

$$\mu(x) = \mathcal{M} + \mathcal{K}x, \qquad (6)$$

$$\Sigma(x) = h + \sum_{k=1}^{n} x_k H_k, \qquad (7)$$

where $h, H_k \in \mathbb{R}^{n\times n}$ for $k = 1, \ldots, n$. The process $N_t \in \mathbb{N}^n$ is a Poisson counting process with arrival intensity $\lambda_t = \lambda_0 + \lambda_1 X_t$ for $\lambda_0 \in \mathbb{R}_+^n$, $\lambda_1 \in \mathbb{R}_+^{n\times n}$, and ξ_t is a vector of jump sizes which distribution $p(\xi)$ is assumed to have a known generating function $\varrho(u) = Ee^{u\xi}$. We assume that the moment generating function exists ($\varrho(u) < \infty$ for $u \in \mathbb{R}^n$).

The affine jump-diffusion process in (5) is a continuous time process. The advantage of the affine class is that it allows for, among other things, processes to be restricted to be positive. In a discrete time model such as the one considered here, it is still permissible to model discrete time decisions that depend on continuously evolving state variables. Decisions then depend on the value of the state variables at discrete times $t = 1, 2, \ldots, \infty$. This effectively eliminates the need to keep track of what the process does between discrete decision times. Alternatively, we may interpret X_t as a process that evolves in discrete time but with the same probability law as the corresponding continuous time model.[3]

2.2. Dividend and Consumption Processes

We assume that the consumption and dividend growth rates are linear functions of the state variables,

$$g_{c,t} = \gamma_c' X_t, \tag{8}$$

$$g_{d,t} = \gamma_d' X_t. \tag{9}$$

This implies that

$$C_{t+s} = C_t \exp\left(\sum_{u=t+1}^{t+s} \gamma_c' X_u\right), \tag{10}$$

$$D_{t+s} = D_t \exp\left(\sum_{u=t+1}^{t+s} \gamma_d' X_u\right). \tag{11}$$

The vectors γ_c and γ_d may contain zeros in such a way that the respective growth rates are driven by specific state variables; i.e., they may be selection vectors.

It is important that the consumption and dividend processes themselves are random walks and hence that the corresponding growth rates are stationary. In some cases, as will be illustrated in the example models, it is convenient to model both consumption and dividends as random walks. In this case, we may transform the state variables by taking first differences of the nonstationary components.

2.2.1. Returns on the Aggregate Wealth. Let z_t denote the log price consumption ratio. We use the Campbell-Schiller approximation to approximate the return on aggregate wealth:

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1}, \tag{12}$$

where κ_0 and κ_1 are linearization constants. Appendix C outlines how to compute these endogenously. The linear form allows us to maintain a tractable analytical form for the pricing kernel. Returns on all other assets will be computed explicitly, without approximation.

We conjecture a solution

$$z_t = A + B' X_t \tag{13}$$

to the log price consumption ratio. Appendix B shows that the Euler equation for aggregate wealth produces the following equations for A and B:

$$0 = \beta\left(1, \theta\left(1 - \frac{1}{\psi}\right)\gamma_c + \theta\kappa_1 B\right) - \theta B, \tag{14}$$

$$A = \frac{\alpha(1, \theta(1-1/\psi)\gamma_c + \theta\kappa_1 B) + \theta(\ln\delta + \kappa_0)}{\theta(1-\kappa_1)}. \tag{15}$$

The solution to these equations can be computed explicitly in a number of cases in which α and β have simple forms. In the case that we can only compute α and β numerically, Equation (14) must be solved numerically. This equation may also have multiple roots. Tauchen (2005) shows that the coefficient corresponding to the stochastic volatility variable(s) in his model solves a quadratic equation and hence has two roots. It is similarly the case in the stochastic volatility examples presented below that the element of $\hat{\beta}$ corresponding to the volatility process has two roots. In the example models there is typically one solution that provides economically reasonable behavior of financial prices and a second solution that does not. For instance, one solution often produces a negative equity premium or an equity premium that is decreasing in risk aversion. Thus, for the example models considered here it is easy to find the only economically reasonable solution to (14).

2.2.2. Prices of Simple Claims. Consider an asset that pays a single dividend, D_{t+s}, at date $t+s$. The value of this claim is

$$\begin{aligned} P_t &= E_t(\delta^\theta C_t^{-\theta/\psi} G_{t+1}^{\theta/\psi} R_{a,t+1}^{-(1-\theta)} E_{t+1} \\ &\qquad \cdot (\delta^\theta G_{t+2}^{-\theta/\psi} R_{a,t+2}^{-(1-\theta)} \cdots E_{t+s-1}(\delta^\theta G_{t+s}^{-\theta/\psi} R_{a,t+s}^{-(1-\theta)} D_{t+s}) \cdots)) \\ &= E_t(\delta^{\theta s} C_t^{-\theta/\psi} G_{t+1}^{-\theta/\psi} G_{t+2}^{-\theta/\psi} \cdots G_{t+s}^{-\theta/\psi} R_{a,t+1}^{-(1-\theta)} \\ &\qquad \cdot R_{a,t+2}^{-(1-\theta)} \cdots R_{a,t+s}^{-(1-\theta)} D_{t+s}) \\ &= E_t(\delta^{\theta s} C_t^{-\theta/\psi} G_{t+1:s}^{-\theta/\psi} R_{a,t+1:s}^{-(1-\theta)} D_{t+s}), \end{aligned}$$

where

$$G_{t+1:s}^{-\theta/\psi} = \prod_{u=t+1}^{t+s} G_u^{-\theta/\psi}, \qquad R_{a,t+1:s}^{-(1-\theta)} = \prod_{u=t+1}^{t+s} R_{a,u}^{-(1-\theta)}.$$

We have used the law of iterated expectations and the Martingale property of the discounted prices and the fact that the return is defined as $R_{t:s} = D_{t+s}/P_t$.

We are interested in the expectations of the form $E_t \exp(\sum_{u=t+1}^{t+s} A_u X_u)$ for $A_u \in \mathbb{R}^n$. The following technical lemma characterizes these expressions explicitly.[4]

[3] Darolles et al. (2006) study continuous time affine processes observed at discrete times and call this class compound autoregressive processes.

[4] This result has been derived independently and in a different context by Darolles et al. (2006). An application to no-arbitrage credit risk can be found in Gourieroux et al. (2006).

Lemma 1. *Let X_t be a process with transition density $p(X_{t+1} \mid X_s)$, the Laplace transform of which is $E(\exp(y'X_{t+s}) \mid X_t = x) = \exp(\alpha(s, y) + \beta(s, y)'x)$. Then for constant vectors $A_1, \ldots, A_s \in \mathbb{C}^n$*

$$E\left(\exp\left(\sum_{u=t+1}^{t+s} A_u' X_u\right) \middle| X_t = x\right) = \exp(\hat{\alpha}(A) + \hat{\beta}(A)x), \quad (16)$$

where $\hat{\beta}$ and $\hat{\alpha}$ are defined through the recursions

$$\begin{aligned}\hat{\alpha}(A) = {} & \alpha(1, A_s) + \alpha(1, A_{s-1} + \beta(A_s)) \\ & + \cdots + \alpha(1, A_{s-1} + \beta(A_{s-1} + \beta(A_s)) \\ & + \cdots + \beta(A_1 + \beta(A_2 + \cdots + \beta(A_{s-1} + \beta(A_s)) \cdots))),\end{aligned} \quad (17)$$

$$\begin{aligned}\hat{\beta}(A) = {} & \beta(1, A_1 + \beta(1, \ldots, A_{s-2} \\ & + \beta(1, A_{s-1} + \beta(1, A_s)) \cdots)).\end{aligned} \quad (18)$$

The following result establishes the foundation for the evaluation of stocks and bonds.

Proposition 1. *Given a consumption process $C_{t+s} = C_t \exp(\sum_{u=t+1}^{t+s} \gamma_c' X_u)$ the price, P_t, of an asset that pays a single dividend claim $D_{t+s} = D_t \exp(\sum_{u=t+1}^{t+s} \gamma_d' X_u)$ is*

$$P_t = D_t \exp(F(s) + \hat{\beta}(\bar{b}_s)' X_t), \quad (19)$$

where

$$F(s) = [\theta \ln \delta - (1-\theta)(\kappa_0 + (\kappa_1 - 1)A)]s + \hat{\alpha}(\bar{b}_s)$$

and the sequence $\bar{b}_s = \{b_u\}_{u=1}^{s+1}$ is given by

$$\begin{aligned} b_1 &= \left(\theta - \frac{\theta}{\psi} - 1\right)\gamma_c - (1-\theta)(\kappa_1 - 1)B + \gamma_d + (1-\theta)\kappa_1 B, \\ b_u &= \left(\theta - \frac{\theta}{\psi} - 1\right)\gamma_c - (1-\theta)(\kappa_1 - 1)B + \gamma_d, \\ & \qquad \text{for } u = 2, \ldots, s \\ b_{s+1} &= -(1-\theta)\kappa_1 B, \end{aligned} \quad (20)$$

and where A and B solve Equations (15) *and* (14), *and the functions $\hat{\alpha}$ and $\hat{\beta}$ are defined in Lemma* 1.

2.2.3. Prices of Bonds and the Term Structure. The explicit expression for a one period discount bond is found by setting $s = 1$ and $D_{t+s} = 1$ in Proposition 1

$$P_t(s) = \exp(F(s) + \hat{\beta}(\bar{b}_s)' X_t), \quad (21)$$

where $F(s)$ and $\hat{\beta}(\bar{b}_s)$ are obtained through Proposition 1 by setting $\gamma_D = 0$. The yield to maturity of an s period zero coupon bond is

$$r_t(s) = -\frac{F(s)}{s} - \frac{\hat{\beta}(\bar{b}_s)'}{s} X_t. \quad (22)$$

Because the yields are just linear combinations of affine processes, they are themselves affine processes. We define the "short rate" to be the annualized yield to maturity of a one-period (one-month) zero coupon bond.

2.2.4. Prices of Stocks. The price of a claim to the perpetual stream D_t obtains as follows.

Proposition 2. *Assume that X_t is stationary. There exists a subjective discount rate $\delta \in (0, 1)$ such that $\sum_{u=t+1}^{\infty} \exp(F(u) + \hat{\beta}(b_u)' X_t) < \infty$. The price of a stock, P_t, that pays a perpetual dividend stream $D_{t+s} = D_t \exp(\sum_{u=t+1}^{t+s} \gamma_d' X_u)$ is then*

$$P_t = D_t \sum_{u=t+1}^{\infty} \exp(F(u) + \hat{\beta}(b_u)' X_t), \quad (23)$$

where the functions F and $\hat{\beta}$ are as in Proposition 1 *and b_u is the sequence $b_u = \{b_v\}_{v=1}^{u+1}$ of constant vectors corresponding to a time u dividend payment*, *as in Proposition* 1.

The above characterization eliminates the need for linearization of the log price-dividend ratios in computing theoretical stock prices. Note that the formula in (23) is not linear in X_t.

3. Empirical Application

We now discuss a specific model and its ability to explain stylized facts from financial market data. We are interested in specifying a model that can capture known asset pricing puzzles such as the equity premium and the risk-free rate puzzles, as well as capturing other dimensions of the data such as stock market crashes. A key point of the model presented here is that it produces an upward-sloping average yield curve. This contrasts with some existing long-run risk models for which the average yield curve is inverted.

Long-run risk models are well known to resolve some of the known asset pricing puzzles. Notably, Bansal and Yaron (2000, 2004) demonstrate that their model generates a significant equity premium and a low real risk-free rate and high-equity volatility yet maintains a consumption growth rate with low volatility. Bansal and Yaron (2000) discuss the slope of the yield curve in their model and note that, on average, the term structure has a negative slope. Although their model is set in real terms, the results from real models carry over to nominal models if one assumes that inflation dynamics does not impact the real growth rates (superneutrality). Because the nominal yield curve in the United States during most time periods has been upward sloping, the inverted yield curve represents a challenge to long-run risk models.

Piazzesi and Schneider (2006) propose a nominal bond pricing model based on long-run risk/Epstein-Zin. Their model contains recursive preferences and inflation and real growth processes that follow a Gaussian vector autoregressive moving average (VARMA)(2,2). Their estimated parameters imply that shocks to inflation have a negative long-run impact on real growth. Thus, in their model, the inflation plays a role similar to that of the current paper in that it correlates negatively with long-run real growth. The two models are still different along some important dimensions: First, the current model has stochastic volatility and jumps. The presence of both these components makes it harder to achieve realistic term structure patterns, as these risks tend to invert the yield curve. Second, although Piazzesi and Schneider (2006) do not consider their model's implication for equity returns, the model considered here generates a large equity premium. Finally, all results in this paper are based on positive subjective discount rates and values of the risk-aversion parameter lower than 10, whereas Piazzesi and Schneider use a negative subjective discount rate ($\delta = 1.005$) and very high risk aversion ($\gamma = 57$) to generate a positively sloping yield curve.

The goal of the example model considered here is to capture the equity premium and the risk-free rate puzzles, generate an upward-sloping yield curve, and capture the possibility departures from normality that have been extensively documented for equity returns. To this end we study a model that generalizes the Bansal and Yaron (2004) long-run risk model to a setting in which the volatility driving consumption and dividend growth rates are affected by possibly large shocks. The dynamics for consumption and dividends are derived from the continuous time model,

$$d\ln C_t^* = (\mu_c + x_t + Q_c m_t - \tfrac{1}{2}V_t)dt + \sqrt{V_t}dB_t^c, \tag{24}$$

$$d\ln D_t^* = (\mu_d + \phi x_t + Q_d m_t - \tfrac{1}{2}\varphi_d^2 V_t)dt + \varphi_d\sqrt{V_t}dB_t^d, \tag{25}$$

where actual consumption C_t and dividends D_t equal the continuous processes $C_t = C_t^*$ and $D_t = D_t^*$ at discrete times. The variable x_t determines the expected real growth rate. The expected inflation, m_t, impacts the real consumption and dividend growth rates, and the impacts are given by the parameters Q_c and Q_d. The parameter μ_c is the unconditional long-term expected consumption growth and ϕ is a "dividend leverage" parameter that, when greater than unity, indicates that corporate dividends can be seen as levered claims to aggregate consumption plus idiosyncratic noise. The terms $\frac{1}{2}V_t$ and $\frac{1}{2}\varphi_d^2 V_t$ are included to ensure that the mean geometric growth rates equal μ_c and μ_d, respectively.

The processes x_t and V_t account for time variation in expected growth and volatility respectively. They follow

$$dx_t = -\kappa_x x_t dt + \varphi_e\sqrt{V_t}dB_t^x, \tag{26}$$

$$dV_t = \kappa_v(\bar{V} - V_t)dt + \sigma_V\sqrt{V_t}dB_t^V + \xi_t dN_t. \tag{27}$$

The parameters κ_x and κ_v measure the speed of mean reversion in expected growth rates and volatility, respectively. The parameter φ_e measures the amount of volatility in the expected growth rate. Innovations in the volatility process V_t come from two sources—shocks in the Brownian motion B^V and the compound Poisson process $\{\xi_t, N_t\}$. The jump sizes, ξ_t, are assumed to be exponentially distributed,

$$\xi_t \sim \exp(\mu_V), \tag{28}$$

and the Poisson process, N_t, has independent arrivals with intensity parameter $\lambda_0 + \lambda_1 V_t$ such that volatility jumps arrive more frequently when volatility is already high if $\lambda_1 > 0$.

The jump in volatility specification allows us to study the equilibrium stock price impacts of small and large shocks to the aggregate volatility process. This is empirically relevant because the volatility process V_t not only impacts the volatility of the macroeconomy but also impacts stock price volatility in our model. A number of papers document a negative correlation between stock price volatility and stock returns. We are interested therefore in the magnitude by which prices depreciate for different volatility shocks in our model. For large volatility shocks, modelled through the jump term $\xi_t dN_t$, we demonstrate that stock prices exhibit crashlike behavior. This is consistent with casual empirical observations; for example, implied volatility of the S&P 500 increased dramatically from about 25% to 70%–80% during the crash of October 19, 1987. More recently, the −3.5% drop in the S&P 500 on February 27, 2007, was accompanied by an increase in the VIX implied volatility index from about 11.5% to 18%. Eraker et al. (2003) and Eraker (2004) find strong evidence for the simultaneous occurrence of volatility jumps and negative price jumps in the context of no-arbitrage models. The model considered here offers an equilibrium interpretation of these results.

Our model, so far, has abstracted from inflation, and prices are interpreted in units of the consumption good. To introduce a framework that rivals that of no-arbitrage models in applicability, we now turn to a model of nominal bond yields and nominal stock returns. Part of what motivates this extension is the question of whether we can specify the model in such a way that we can recover a positively sloping yield curve while maintaining essential model characteristics that make the models consistent with observed equity market data.

In pursuing this we will model inflation as exogenously given. If inflation is an exogenously given random walk independent of the real variables in the economy (superneutral), it is straightforward to show that inflation has no effect on prices. Thus, under inflation superneutrality the nominal term structure equals the real term structure plus expected inflation growth. Inflation neutrality, therefore, does not turn the negatively sloping real yield curve of Bansal and Yaron (2000) into a positively sloping nominal one.

We specify a model in which inflation and long-term inflation growth affect financial market prices. We are particularly interested in analyzing a situation in which long-term inflation growth correlates negatively with real growth rates. This is consistent with the empirically observed strong negative correlation of −0.52 between inflation and real growth rates in consumption and dividends. It is also consistent with empirical evidence in Christiano et al. (1999), Ang and Piazzesi (2003), and Piazzesi and Schneider (2006) among others, showing negative impulse response functions for monetary shocks on real aggregates.

In considering the impact on inflation on nominal zero coupon bond prices, consider a general formulation where inflation is a linear function of state-variables,

$$i_t = \gamma_I' X_t.$$

The nominal price of a \$1 zero coupon bond is then

$$P_t(s) = E_t \exp\left(s\theta \ln\delta - \frac{\theta}{\psi} g_{t+1:t+s} - (1-\theta) r_{a,t+1:t+s} - i_{t+1:t+s}\right). \quad (29)$$

This expression is identical to the expression for a single dividend paying stock with $\gamma_D = -\gamma_I$. Thus, we can interpret nominal bond prices in the presence of inflation as a claim to (negative) future inflation growth.

We assume the following inflation dynamics:

$$di_t = m_t dt + \sigma_i dB_t^i, \quad (30)$$

$$dm_t = \kappa_m(\bar{i} - m_t)dt + \sigma_m \sqrt{m_t} dB_i^m, \quad (31)$$

where i_t is the inflation rate and m_t is the expected inflation rate.

We calibrate the nominal price model using inflation parameters $\bar{i} = 0.0033$, $\kappa_m = 0.03$, $\sigma_I = 8e-6$, and $\sigma_m = 0.01$. This generates an annual average inflation of 4%, with a standard deviation of 3.8% and first-order autocorrelation of 0.7. These numbers are chosen to match the postwar inflation data in which the correlations between real dividend growth and consumption growth and inflation both are about −0.52. In addition, we choose $Q_c = -0.65$ and $Q_d = \phi Q_c$, respectively. This generates annual correlations of −0.52 and −0.48 between inflation and consumption and inflation and consumption and dividend growth, respectively. This compares to −0.52 in the data.

Table 1 presents key asset price moments, including bootstrapped confidence intervals. The table serves as a benchmark that the theoretical models can be held to. The stock price data are based on the monthly Fama-French data from 1926–2006. The yield curve data is from Gurkaynak et al. (2006), using data from 1971–2006. The tabulated sample moments and corresponding confidence intervals in Table 1 suggest that key moments such as the equity premium and the average yield on one- and ten-year bonds are imprecisely estimated. For example, the 98% confidence interval for the equity premium is (2.23, 12.29), which is quite wide.

Table 1 Moments of Asset Returns Data

	1 Equity prem.	2 Ret. std.	3 Skew.	4 Kurt.	5 1 yr.	6 Std.	7 10 yr.	8 Std.
	7.83	18.78	0.219	10.93	6.57	2.96	7.63	2.34
1%	2.23	14.38	−1.04	4.36	4.77	1.5	6.26	1.04
99%	12.09	24.84	1.19	13.89	9.06	3.94	9.85	3.03

Notes. This table reports key moments of U.S. asset market data. The equity premium is the average rate of return over the risk-free rate using the CRSP value weighted index and data from 1926 to 2006. Columns 2–4 show the standard deviation, skewness, and kurtosis of market returns. Columns 5–8 show average and standard deviations for nominal bond yields of maturities for 1 and 10 years using data from Gurkaynak et al. (2006) from 1971 to 2006. The bottom two rows give bootstrapped 1 and 99 percentiles of the sampling distributions.

Table 2 displays the main asset pricing characteristics of the nominal pricing model. First, the results show that the model is capable of generating equity premiums from virtually zero to more than 9% for $\gamma = 8$ and $\psi = 5$. These preference parameters are within the range many economists would consider reasonable. Thus, this model, like other long-run risk models, resolves the equity premium puzzle. It also generates low-risk free rates for larger values of ψ. If ϕ is equal to one half, the model generates a small equity premium and a very high-risk free rate. Because the standard CRRA power utility model obtains as a special case of Epstein-Zin when $\psi = 1/\gamma$, low values of ψ produce asset prices that resemble prices in the CRRA model. It is not surprising, therefore, that low values of ψ fail to resolve the equity premium and risk-free rate puzzles.

Table 2 also illustrates the interaction between asset prices and expected inflation. The superneutral case ($Q_c = Q_d = 0$) produces two notable effects: First, the equity premium is somewhat lower than in the nonneutral case. Second, the super-neutral model produces a downward sloping nominal yield curve. In contrast, the nonneutral model produces a positively sloping yield curve for high values of the IES parameter, ψ. This is illustrated in Figure 1, which shows that the yield curve is always downward sloping for the superneutral parameter configuration. Conversely, when expected inflation negatively impacts real growth, the yield curve is upward sloping for

Table 2 Asset Price Implications of the Nominal Price Model

γ	6			8			10		
ψ	0.5	3	5	0.5	3	5	0.5	3	5
Nonneutral									
Equity prem.	0.53	2.46	2.70	1.78	5.33	5.67	3.88	8.60	9.13
Mean (annual)	11.33	9.52	9.44	13.10	12.18	12.07	16.06	15.14	15.20
Std. (annual)	11.69	19.52	20.33	11.64	18.98	19.58	12.27	18.72	19.24
Skewness	−0.04	−0.03	−0.03	−0.05	−0.03	−0.03	−0.04	−0.03	−0.03
Kurtosis	3.29	3.36	3.37	3.30	3.35	3.35	3.36	3.37	3.38
Corr($\triangle V_t$, r_t)	−0.02	−0.03	−0.03	−0.05	−0.06	−0.06	−0.07	−0.07	−0.07
1-yr. bond yield	10.80	7.06	6.74	11.31	6.85	6.41	12.18	6.54	6.07
10-yr. bond yield	9.81	7.35	7.19	9.57	7.49	7.26	9.76	7.77	7.68
1-yr. yield std.	3.56	1.80	1.89	3.58	1.72	1.79	3.65	1.62	1.66
10-yr. yield std.	1.31	0.69	0.73	1.33	0.76	0.82	1.36	0.91	0.98
Superneutral									
Equity prem.	0.07	1.18	1.33	0.67	2.46	2.65	1.17	3.68	3.72
Mean (annual)	10.69	8.49	8.35	11.46	9.69	9.56	12.20	10.85	10.55
Std. (annual)	10.43	15.89	16.37	10.24	15.25	15.67	10.01	14.66	15.07
Skewness	−0.00	−0.00	−0.01	−0.01	−0.01	−0.02	−0.01	−0.04	−0.04
Kurtosis	3.32	3.36	3.36	3.35	3.38	3.40	3.46	3.54	3.51
Corr($\triangle V_t$, r_t)	−0.02	−0.04	−0.04	−0.06	−0.09	−0.09	−0.10	−0.13	−0.13
1-yr. bond yield	10.62	7.31	7.02	10.79	7.23	6.91	11.02	7.17	6.83
10-yr. bond yield	9.64	6.94	6.72	9.16	6.67	6.46	8.70	6.36	6.17
1-yr. yield std.	4.20	2.51	2.45	4.28	2.52	2.46	4.32	2.57	2.45
10-yr. yield std.	1.47	0.79	0.76	1.50	0.80	0.77	1.52	0.82	0.77

Notes. This table reports key asset price properties for nominal financial prices for different values of intertemporal elasticity of substitution, ψ, and timing resolution of uncertainty, γ. The numbers are population moments. We consider two inflationary regimes: The first is superneutral inflation, which implies no correlation between expected inflation shocks and real growth, $Q_c = Q_d = 0$. The second is a nonneutral case where expected inflation impacts long-run real growth through $Q_c = -0.65$ and $Q_d = -2.275$. The remaining parameters are $\kappa_x = 0.025$, $\mu = \mu_d = 0.0015$, $\phi = 3.5$, $\bar{v} = 0.02^2/12$, $\kappa_v = 0.04$, $\varphi_e = 0.07$, $\varphi_d = 4.5$, $\sigma_v = 2.2e-4$, $\lambda_0 = 0$, $\lambda_1 = 252$, $\mu_V = 2e-5$, $\bar{i} = 0.04/12$, $\kappa_m = 0.03$, $\sigma_I = 8e-6$, $\sigma_m = 0.01$.

$\psi = 3$ and $\psi = 5$. Low values of ψ produce negative term premia even for the nonneutral model.

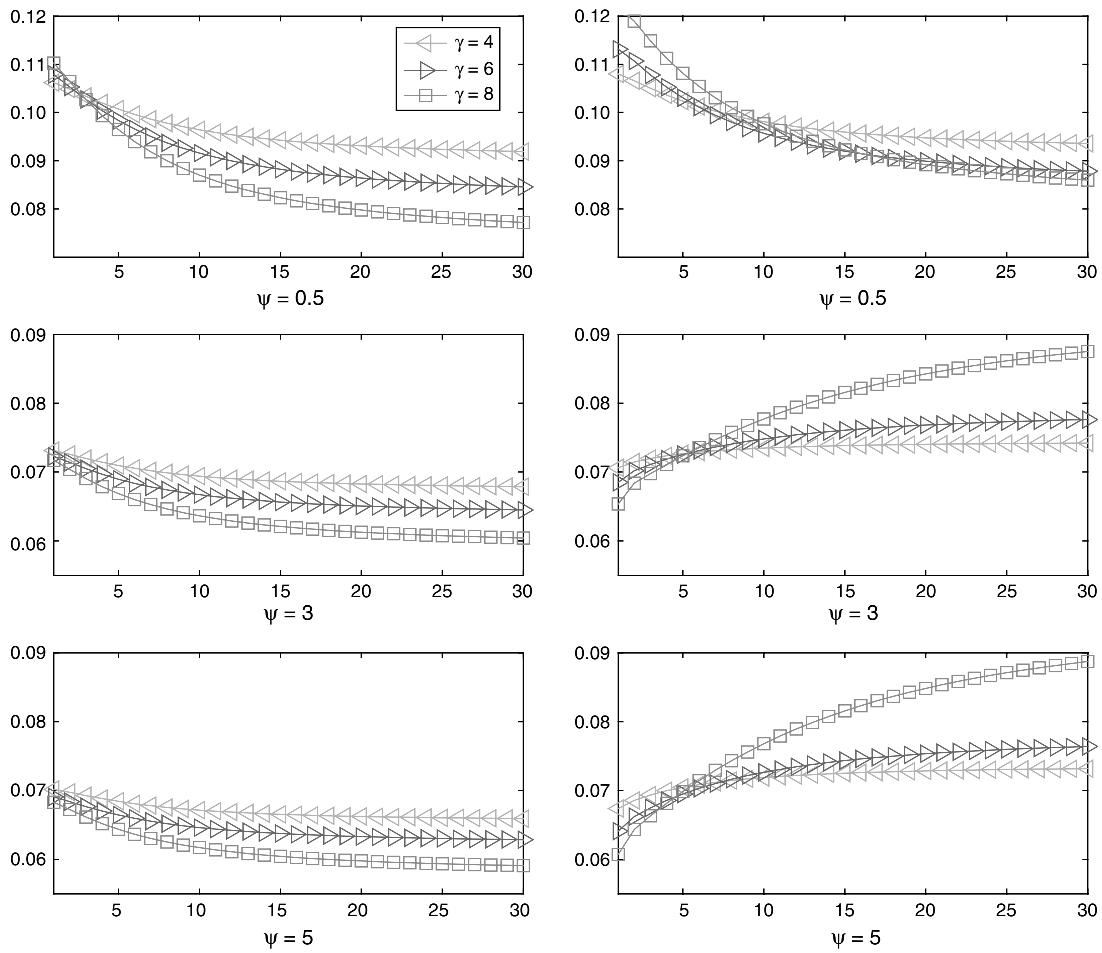
3.1. Impact of Jumps

A novel feature of the model considered here is the possibility of large increases in macroeconomic uncertainty, as suggested by the presence of jump component in the volatility. We investigate the impact of these jumps on asset prices in Figure 2, which shows the impulse response of comparatively large volatility shocks for the models with and without volatility jumps. A volatility shock with a one percent chance of occurring leads to a negative 4.7% drop in the log stock price for the jump model and a 2.5% drop in the model with the continuous volatility path. Thus, the inclusion of volatility jumps in the model implies much more pronounced negative stock price reactions to shocks in the economic uncertainty.

Table 3 tabulates the pricing implications for a model *without* volatility jumps. This allows us to compare the effects of including jumps in the model, as shown in Table 2. To make a fair comparison, we ensure that $\bar{v}$ and σ_v are chosen such that the first two moments of V_t match the case with jumps. This allows us to isolate the equilibrium effects of introducing jumps.

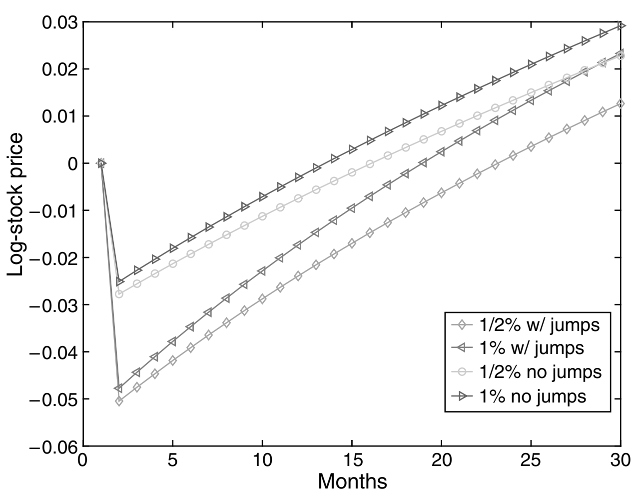
The following patterns emerge: First, the equity premiums are systematically lower, by as much as one percent, when excluding the jump component. Second, the correlation between stock returns and changes in volatility is more negative for the jump models in Table 2 than the nonjumping models in Table 3. Third, the yield curve has a more pronounced negative slope when there are jumps in the model. This is related to the following economic effect: A shock in volatility makes the stock market relatively unattractive, and prices fall. Bond prices respond positively, and yields negatively, in response to a volatility shocks. The bond "factor loadings" are greater in magnitude for longer-maturity bonds. They are also greater in magnitude when the model includes jumps. This is illustrated in Figure 3, which shows the bond factor loadings as a function of maturity with and without jumps in the model. Thus, the longer-maturity bonds carry a larger volatility risk premia for the jump model than the nonjump model. Because the volatility risk premia are negative, the presence of jumps in the model leads to a more inverted yield curve than in the pure diffusion model. More generally, the more "action" in the stochastic volatility process, the more inverted the yield curve. For this reason, it is much more challenging to formulate a long-run risk model that produces an upward sloping yield curve in an

Figure 1 Nominal Term Structures: (Left) Super-Neutral Case; (Right) Nonneutral Case



environment *with* stochastic volatility than to formulate one in an environment without. We illustrate this again in Figure 4, which shows that yield curve turns increasingly negatively sloping as the volatility of σ_v increases. This effect is also present and much stronger for volatility jump size μ_V.

Figure 2 Impulse-Response Functions Stock Prices Given Shocks to the Volatility Process



Notes. Shown is the impact on the stock price of negative shock with a 0.5% and 1% chance of occurring, and for the model with and without jumps in volatility.

Note that because the model in Table 3 contains no jumps and produces a zero (expected) inflation risk premium, it has only two priced risk factors, x_t and V_t. Nominal prices in this model equal real prices plus expected inflation growth. The real prices produced by this model are therefore very similar to those of the Bansal and Yaron (2004) model, and the numbers displayed are in the same ballpark as those reported by Bansal and Yaron (2004). Clearly, adding the risk premium for the expected inflation as well as volatility jumps leads to a dramatic increase in the equity premium. Alternatively, by including equilibrium inflation and jump risk premiums, the model is capable of capturing the equity premium, risk-free rate, and term premium at lower values of γ than in Bansal and Yaron (2004) and at much lower values than in Piazzesi and Schneider (2006).

3.2. Euler Equation Errors

In a recent paper, Lettau and Ludvigson (2005) argue that "leading asset pricing models" do not fully

Table 3 Asset Price Implications Without Jumps

γ	6			8			10		
ψ	0.5	3	5	0.5	3	5	0.5	3	5
Nonneutral									
Equity prem.	0.50	2.23	2.43	1.61	3.06	4.50	3.66	7.68	8.02
Mean (annual)	11.30	9.39	9.24	12.87	8.77	7.55	15.71	14.43	14.33
Std. (annual)	10.81	18.27	18.98	10.86	16.43	11.93	11.66	17.36	17.82
Skewness	−0.04	−0.04	−0.03	−0.06	−0.04	−0.05	−0.05	−0.02	−0.02
Kurtosis	3.22	3.33	3.35	3.28	3.39	3.24	3.36	3.39	3.39
Corr($\triangle V_t, r_t$)	−0.01	−0.02	−0.02	−0.03	−0.03	0.01	−0.04	−0.04	−0.04
1-yr. bond yield	10.79	7.16	6.81	11.26	5.71	3.04	12.05	6.75	6.31
10-yr. bond yield	10.00	7.52	7.31	9.86	5.58	2.58	10.17	7.63	7.43
1-yr. yield std.	3.16	1.78	1.86	3.14	1.65	2.87	3.22	1.53	1.49
10-yr. yield std.	1.16	0.68	0.72	1.16	0.63	1.05	1.19	0.78	0.81
Superneutral									
Equity prem.	0.22	0.86	1.12	0.60	1.92	2.08	0.90	2.92	3.10
Mean (annual)	10.83	8.24	8.21	11.34	9.23	9.09	11.81	10.20	10.06
Std. (annual)	9.27	14.10	14.53	9.16	13.62	14.09	8.92	13.23	13.72
Skewness	0.01	0.00	0.00	0.00	0.01	0.00	0.00	−0.00	−0.01
Kurtosis	3.21	3.21	3.22	3.20	3.21	3.22	3.21	3.21	3.18
Corr($\triangle V_t, r_t$)	−0.01	−0.02	−0.03	−0.04	−0.06	−0.06	−0.06	−0.08	−0.09
1-yr. bond yield	10.61	7.38	7.09	10.74	7.31	7.01	10.90	7.28	6.96
10-yr. bond yield	9.83	7.06	6.83	9.46	6.86	6.64	9.09	6.66	6.46
1-yr. yield std.	3.87	2.42	2.40	3.92	2.49	2.41	3.99	2.50	2.43
10-yr. yield std.	1.34	0.76	0.75	1.36	0.78	0.75	1.39	0.79	0.76

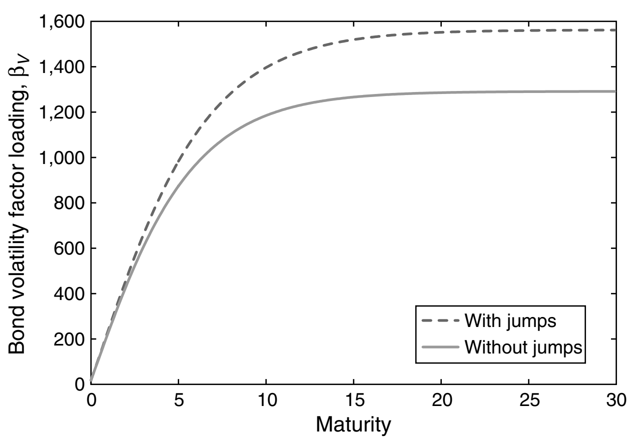
Notes. This table reports implications for nominal financial prices without jumps ($\lambda_0 = \lambda_1 = 0$). All parameters are as in Table 2 except $\bar{v}$ and σ_V, which are $\bar{v} = 2.963e-5$ and $\sigma_V = 3.852e-4$. This ensures the first two conditional moments of the volatility process, V_t, are the same. Remaining parameters are as in Table 2.

capture the failure of the standard consumption-based power utility model. Their argument is simple: The standard power utility C-CAPM Euler equation does not hold empirically for U.S. postwar data. That is, there exist no values of γ that will set the average pricing error

$$e = E(\exp(-\gamma^* g)R)$$

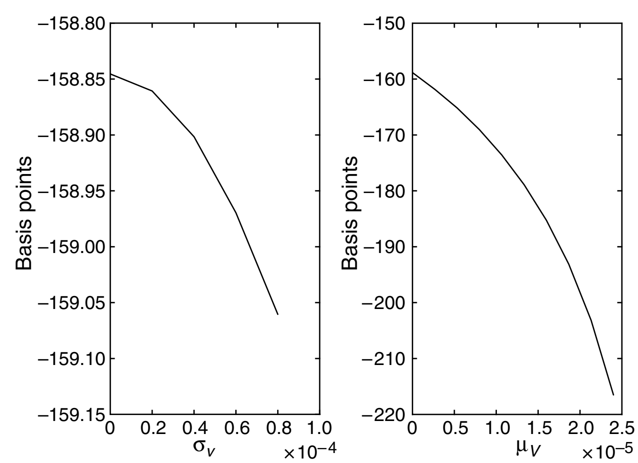
to zero. Here R is a vector (real) of stock returns in excess of the (real) risk-free rate, and g is real log consumption growth, and γ^* minimizes e^2. Assume that some model is the "true" model of asset price dynamics, meaning that the model has the same data-generating process as in the observed data. Thus, if $|e| > 0$ in the data, it must be that $|e| > 0$ for simulated model data as well. Otherwise, the data-generating processes cannot be the same in the model economy as in the observed data.

Figure 3 Volatility Factor Loading for Bonds, β_V, Which Measures the Sensitivity of Interest Rates with Respect to Changes in Macrovolatility, V_t



Note. The factor loadings are shown for the volatility jump model and the model without jumps.

Figure 4 Yield Curve Slopes



Notes. This figure illustrates how the slope of the yield curve (defined as the difference between 30- and 1-year nominal bond yields) depends on the size of exogenous shocks to macro volatility. When "small" (Brownian motion) shocks are magnified through higher values of σ_V (left panel) or large shocks (jump sizes) through μ_V (average jump sizes) (right panel), the yield curve slope becomes increasingly negative.

Table 4 Euler Equation Errors

γ	4	6	8	10
$\psi=3$				
Jumps	0.00	0.42	0.33	0.46
No jumps	0.00	0.00	0.32	0.42
$\psi=5$				
Jumps	0.04	0.27	0.37	0.60
No jumps	0.00	0.20	0.30	0.39

Notes. This table shows Euler equation errors (EEE) for the model with and without jumps in the volatility. Lettau and Ludvigson (2005) compute the EEE for a number of models proposed in the literature and show that these models generate zero EEEs. The EEEs are 0.48 in the Lettau-Ludvigson sample period.

Lettau and Ludvigson (2005) compute e for a number of candidate asset pricing models and show that all the models considered produce zero pricing errors. This contrasts with the data, for which the errors are substantial. In conclusion, therefore, the DGP of observed U.S. consumption and returns data differ from that of "leading models," including the Campbell and Cochrane (1999) external habit model, Menzly et al. (2004) habit model, Guvenen's (2003) limited participation model, and Bansal and Yaron's (2004) long-run risk model.

Lettau and Ludvigson (2005) further demonstrate that positive Euler equation errors can be explained by non-Gaussian innovations in consumption growth and returns. In particular, they show that when data in a limited participation model are generated from (severely) non-Gaussian joint distributions, simulated data can produce Euler errors that match those of observed data. An interesting question, therefore, is whether the non-Gaussian jump models considered here can similarly generate non-zero Euler errors.

Table 4 reports Euler equation errors for the models under consideration here with parameter values as before and with $\psi=3$ and $\psi=5$. As shown, the models generate non-zero errors for a number of the parameter constellations. For actual U.S. consumption and asset returns data, Lettau and Ludvigson (2005) show that the Euler error is 0.48. Table 4 shows that $\gamma=10$ produces a Euler equation error of 0.46 for the model with volatility jumps and 0.42 for the model without. Thus, the volatility jumping model provides a pricing error that is almost identical to that observed in the data. It should also be mentioned that Euler errors computed under these models have very large amounts of statistical estimation uncertainty in small samples.

4. Conclusion

This paper has presented a general framework for valuation of stocks and bonds based on an Epstein-Zin preference structure and under general assumptions about the dynamics of state variables that affect consumption and dividend growth. The example model presented demonstrates that this framework is capable of explaining well-known asset pricing puzzles and generating prices the distributional characteristics of which mimic those observed in equity markets. The Epstein-Zin preference structure represents an important component in explaining asset pricing puzzles as it disentangles the elasticity of substitution from the temporal resolution of uncertainty. This produces low bond yields, high equity returns, and higher-order moments that are in line with observed data.

This paper delivers a framework for analyzing stock and bond prices that effectively can be seen as an equilibrium version of no-arbitrage factor models. The advantages relative to standard no-arbitrage models are that the links to macroeconomic time series are explicit and the factor "loadings" that determine the various assets' sensitivity to changes in the economic variables are explicit function parameters that determine the dynamic behavior of macro quantities and, more importantly, preferences. This allows a fairly rich framework for analyzing the link between macro and financial market variables. It also allows for a fair amount of flexibility in allowing for unobserved components such as expected growth rates and volatility. This is similar in spirit to no-arbitrage models but has the advantage that these latent factors have economic interpretations. Again, this facilitates a closer examination of the links between financial market dynamics and the macroeconomy. In building on this framework it is possible to construct quite flexible models of yield curve dynamics while maintaining an equilibrium foundation. This is likely to produce an interesting equilibrium-based alternative to the growing body of papers that study the link between macroeconomic dynamics and the term structure.

Acknowledgments

The author thanks two anonymous referees and an associate editor for helpful comments. The author also thanks Ravi Bansal, Tim Bollerslev, Michelle Connolly, Ron Gallant, George Tauchen, and participants at the 2006 Econometric Society Summer Meeting, Duke Econometrics workshop, Fuqua brownbag, and the University of Maryland for helpful comments. Bjørn Eraker's current affiliation is Wisconsin School of Business, Madison, Wisconsin 53607.

Appendix A. Recovering α and β

The conditional Laplace transform is exponential affine, as in Equation (4) with coefficients α and β that satisfy the complex Ricatti ODEs

$$\frac{\partial\beta(s,u)}{\partial s} = \mathcal{K}'\beta(t,u)+\tfrac{1}{2}\beta(s,u)'H\beta(s,u) + \lambda_1(\varrho(\beta(s,u))-1), \tag{A1}$$

$$\frac{\partial\alpha(s,u)}{\partial s} = \mathcal{M}'\beta(s,u)+\tfrac{1}{2}\beta(s,u)'h\beta(s,u) + \lambda_0(\varrho(\beta(s,u))-1), \tag{A2}$$

with boundary conditions $\beta(0, u) = u$, $\alpha(0, u) = 0$. The product $\beta(s, u)'H\beta(s, u)$ denotes the n dimensional vector with k'th element $\beta(s, u)'H_k\beta(s, u)$.

Appendix B. Computing A and B

Using the log-linear form of the consumption wealth ratio, the Euler equation becomes

$$
\begin{aligned}
0 &= \ln E_t \exp\left(\theta \ln\delta - \frac{\theta}{\psi} g_{t+1} + \theta r_{a,\,t+1}\right)\\
&= \ln E_t \exp\left(\theta \ln\delta - \theta\left(\frac{1}{\psi} - 1\right) g_{t+1} + \theta\kappa_0 + \theta\kappa_1 z_{t+1} - \theta z_t\right)\\
&= \ln E_t \exp\left(\theta \ln\delta - \theta\left(\frac{1}{\psi} - 1\right)\gamma_c' X_{t+1}\right.\\
&\qquad \left. + \theta\kappa_0 + \theta\kappa_1(A + B'X_{t+1}) - \theta(A + B'X_t)\right)\\
&= \ln E_t \exp\left(\theta \ln\delta + \left(\theta\left(1 - \frac{1}{\psi}\right)\gamma_c + \theta\kappa_1 B\right)' X_{t+1}\right.\\
&\qquad \left. + \theta(\kappa_0 - A) + \theta\kappa_1 A - \theta B'X_t\right)\\
&= \theta\ln\delta + \theta\kappa_0 + \theta(\kappa_1 - 1)A + \alpha\left(1, \left(\theta\left(1 - \frac{1}{\psi}\right)\gamma_c + \theta\kappa_1 B\right)\right)\\
&\qquad + \left[\beta\left(1, \left(\theta\left(1 - \frac{1}{\psi}\right)\gamma_c + \theta\kappa_1 B\right)\right) - \theta B\right]' X_t.
\end{aligned}
$$

The last line uses the expression for $E_t\exp(BX_{t+s})$ in Equation (4). The last line can only be equated to zero if A and B satisfy Equations (15) and (14).

Appendix C. Linearization Constants

The following equations are solved as part of the equilibrium:

$$
\kappa_1 = \frac{e^{E(z_t)}}{1 + e^{E(z_t)}}, \tag{C1}
$$

$$
\kappa_0 = -\ln[(1 - \kappa_1)^{1-\kappa_1}\kappa_1^{\kappa_1}], \tag{C2}
$$

where z_t is the log consumption-wealth ratio.

Appendix D. Proofs

Proof of Lemma 1. Using iterated expectations,

$$
\begin{aligned}
&E_t\left(\exp\left(\sum_{u=t+1}^{t+s} A_u' X_u\right)\right)\\
&= E_t(e^{A_{t+1}'X_{t+1}} E_{t+1}[e^{A'X_{t+2}} \times \cdots \times E_{t+s-1}[e^{A_{t+s}'X_{t+s}}]\cdots])\\
&= E_t(e^{A_{t+1}'X_{t+1}} E_{t+1}[e^{A_{t+2}'X_{t+2}} \times \cdots\\
&\qquad \times E_{t+s-2}[e^{A_{t+s-1}'X_{t+s-1}} E_{t+s-1}[e^{A_{t+s}'X_{t+s}}]]\cdots])\\
&= E_t(e^{A_{t+1}'X_{t+1}} E_{t+1}[e^{A'X_{t+2}} \times \cdots\\
&\qquad \times E_{t+s-2}[e^{\alpha(1,\, A_{t+s}) + (A_{t+s-1} + \beta(1, A_{t+s}))'X_{t+s-1}}]\cdots])\\
&= E_t(e^{A_+{t+1}'X_{t+1}} E_{t+1}[e^{A'X_{t+2}} \times \cdots\\
&\qquad \times\ e^{\alpha(1,\, A_{t+s})} E_{t+s-2}[e^{(A_{t+s-1} + \beta(1,\, A_{t+s}))'X_{t+s-1}}]\cdots])\\
&= E_t(\cdots \times E_{t-s-3}\\
&\cdot[e^{\alpha(1,\, A_{t+s}) + A_{t+s-2}'X_{t+s-2} + \alpha(1,\, A_{t+s-1} + \beta(1,\, A_{t+s})) + \beta(1,\, A_{t+s-1} + \beta(1,\, A_{t+s}))'X_{t+s-1}}]\cdots)\\
&= E_t(\cdots \times e^{\alpha(1,\, A_{t+s}) + \alpha(1,\, A_{t+s-1} + \beta(1,\, A_{t+s}))}\\
&\qquad \cdot E_{t-s-3}[e^{(A_{t+s-2} + \beta(1,\, A_{t+s-1} + \beta(1,\, A_{t+s})))'X_{t+s-1}}]\cdots). \quad \square
\end{aligned}
$$

Proof of Proposition 1.

$$
\begin{aligned}
P_t/D_t &= E_t \exp\left(s\theta\ln\delta - \frac{\theta}{\psi} g_{t+1:s} - (1-\theta) r_{a,\,t+1:s} + \ln D_{t+s}\right)\\
&= E_t \exp\left(s\theta\ln\delta + \left(\gamma_d - \frac{\theta}{\psi}\gamma_c\right)' \sum_{u=t+1}^{t+s} X_u - (1-\theta)\right.\\
&\qquad \left. \cdot \sum_{u=t+1}^{t+s} (\kappa_0 + \kappa_1 A + \kappa_1 B'X_{u+1} - A - B'X_u + \gamma_c' X_u)\right)\\
&= E_t \exp\left([\theta\ln\delta - (1-\theta)(\kappa_0 + (\kappa_1 - 1)A)]s\right.\\
&\qquad + \left(\gamma_d - \frac{\theta}{\psi}\gamma_c\right)' \sum_{u=t+1}^{t+s} X_u - (1-\theta)\\
&\qquad \cdot \sum_{u=t+1}^{t+s} (\kappa_1 B'X_u - B'X_u + \gamma_c' X_u)\\
&\qquad \left. - \kappa_1 B'X_{t+1} + \kappa_1 B'X_{t+s+1}\right)\\
&= E_t \exp\left([\theta\ln\delta - (1-\theta)(\kappa_0 + (\kappa_1 - 1)A)]s\right.\\
&\qquad + \left[\left(\theta - \frac{\theta}{\psi} - 1\right)\gamma_c + \gamma_d - (1-\theta)(\kappa_1 - 1)B\right]'\\
&\qquad \left. \cdot \sum_{u=t+1}^{t+s} X_u - (1-\theta)(\kappa_1 B'X_{t+s+1} - \kappa_1 B'X_{t+1})\right)
\end{aligned}
$$

This is of the form given in Lemma 1 with coefficients as suggested in the proposition. $\square$

Proof of Proposition 2. Stationarity of X_t implies that $(\alpha(s), \beta(s)) \to (\alpha, 0)$ as $s \to \infty$; i.e., the limiting generating function does not depend on the initial state, X_t. This implies that for finite constants $a, b \in \mathbb{R} \times \mathbb{R}^n$

$$
\lim_{s\to\infty} E_t e^{\sum_u^{t+s} b_u' X_u} = e^{as + sb'X_t},
$$

with b_u as given in Proposition 1. Thus,

$$
\begin{aligned}
&\sum_{s=1}^{\infty} e^{[\theta\ln\delta - (1-\theta)(\kappa_0 + (\kappa_1 - 1)A)]s + \hat{\alpha}_s + \hat{\beta}_s' X_t}\\
&\qquad = \sum_{s=1}^{\infty} e^{[\theta\ln\delta^* - (1-\theta)(\kappa_0 + (\kappa_1 - 1)A) + a + b'X_t]s} < \infty
\end{aligned}
$$

for some $\delta^* < \exp\{[(1-\theta)(\kappa_0 + (\kappa_1 - 1)A) + a + b'X_t]/\theta\}$. $\square$

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